Stochastic Integer Programming

by

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(Relatively) New field

Stochastic programming:

Seminal papers : G. Dantzig , Mgt. Sc. (55)
A. Charnes, W. Cooper, G. Symonds , Mgt. Sc. (58)
….
R. Van Slyke, R. Wets SIAM J. A.M. (69)

Stochastic Integer programming:

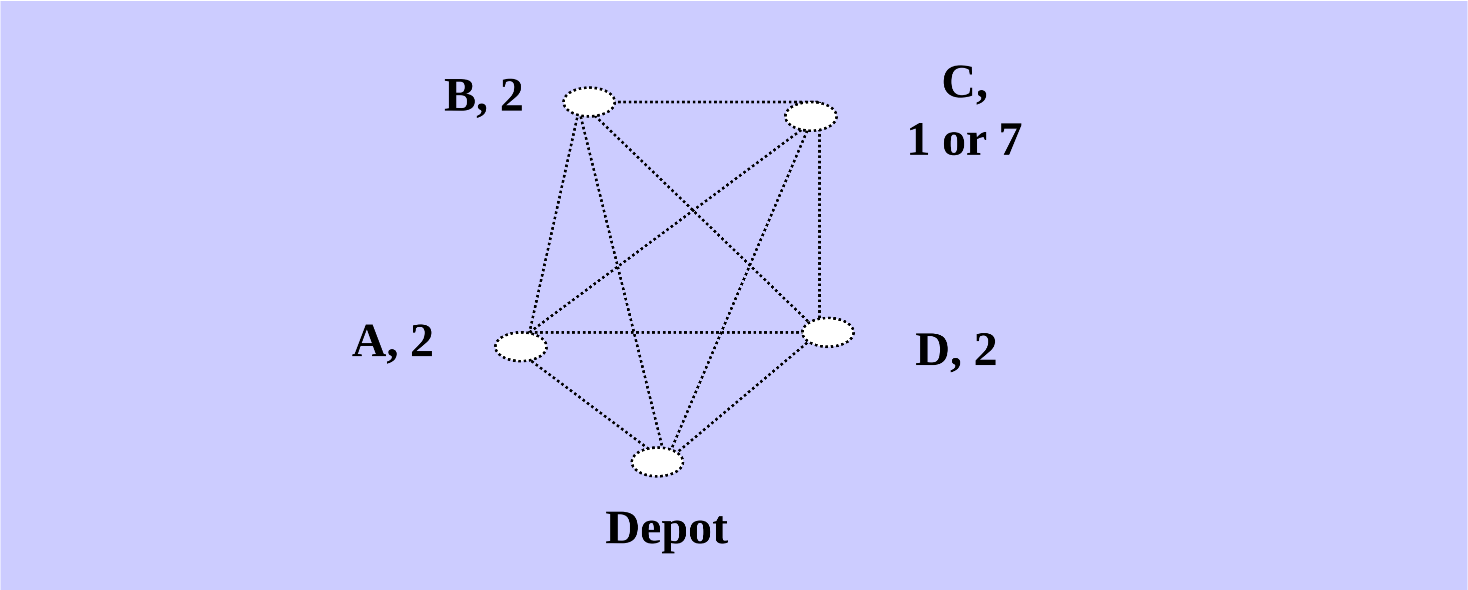
First paper (TTBMK) : 0/1 in the first-stage only : R. Wollmer, M.P. (80)
Recourse function +Asymptotic analysis: L.Stougie (Thesis 87)
….
Integer L-shaped method : G. Laporte, F. Louveaux , ORL (93)
Simple integer recourse: M. Van der Vlerk (Thesis 95)
….

more than 5 sessions with SIP in this conference

Presentation outline

- **Modelling**
- **Difficulty**
- **Exact Methods**
  - **Simple Integer**
  - **Integer L-Shaped (finiteness in first-stage)**
  - **Finiteness / Branching (in the second-stage)**
  - **Reformulation / Valid Inequalities (in the second-stage)**
- **Sampling**
- **Conclusion**

Importance of Uncertainty : New decisions

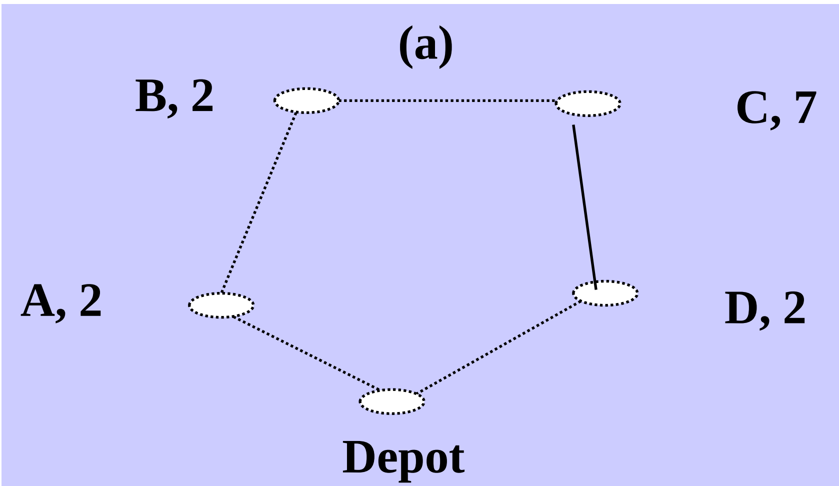


- Vehicle of capacity = 10
- Demand is 2 at nodes A,B,D
- Demand is random at node C:
  1 or 7 with equal probability ½
- No limit on travel time

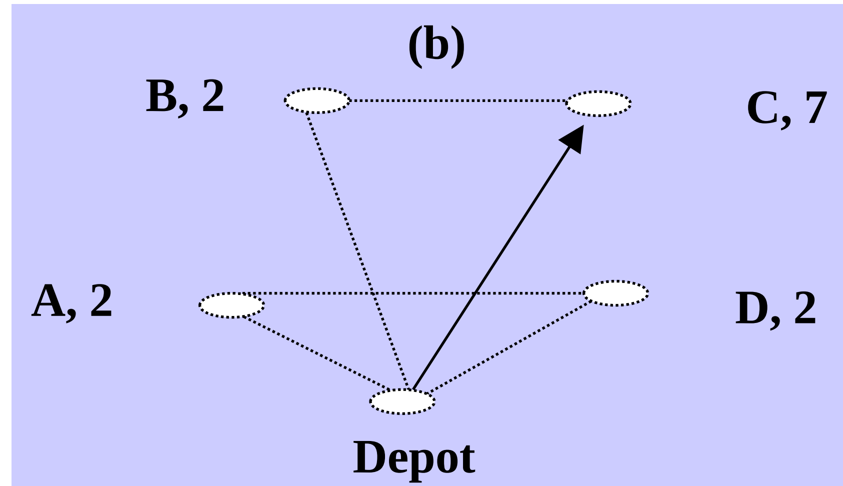
Dist	0	A	B	C	D
0	-	2	4	4	1
A	2	-	3	4	2
B	4	3	-	1	3
C	4	4	1	-	3
D	1	2	3	3	-

« Early information »

Assume we can get the information in advance



Effective length = 10
« demand in C = 1 »



Effective length = 14
« demand in C = 7 »

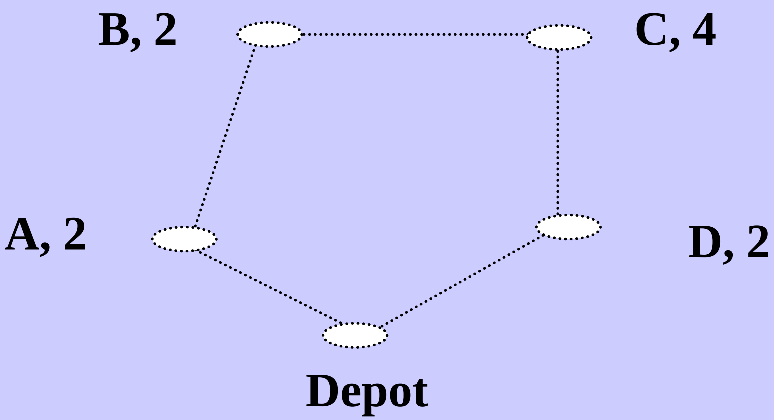
→ **WS = 12.0** = expected length, if information is available beforehand

« Classical » approach : expected value problem

« Forget Uncertainty »: Mean value problem

(or Expected value problem)

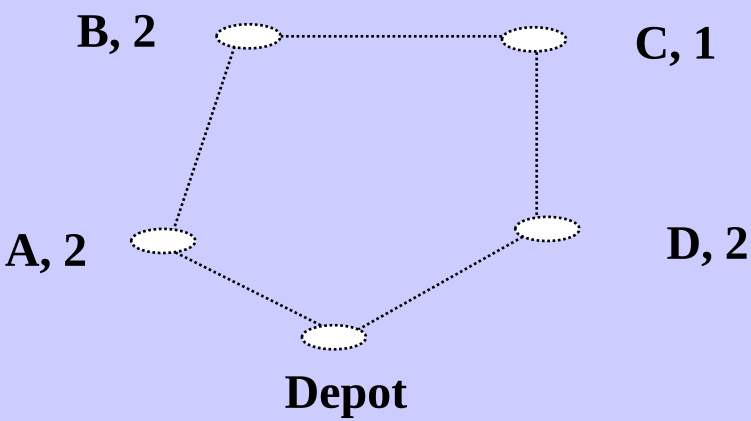
Replace the random variable by its **expectation**



Expected demand in C = 4 :
All demand can be accomodated in one vehicle

« Optimal » Tour : A,B,C,D
Length : 10

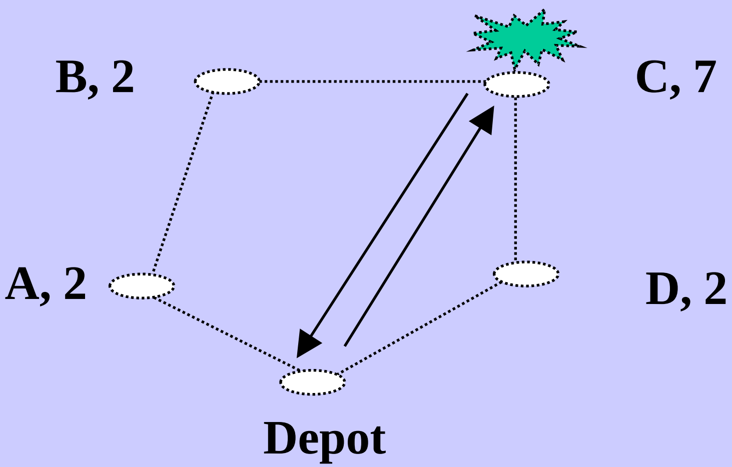
Uncertainty does not forget you !



a. « Demand in C = 1 »

Tour : A,B,C,D **as planned**

Length : 10



b. « Demand in C = 7 »
Failure

Tour:
A,B,C,Depot,C,D

Length : 18

Expected effective length = 14

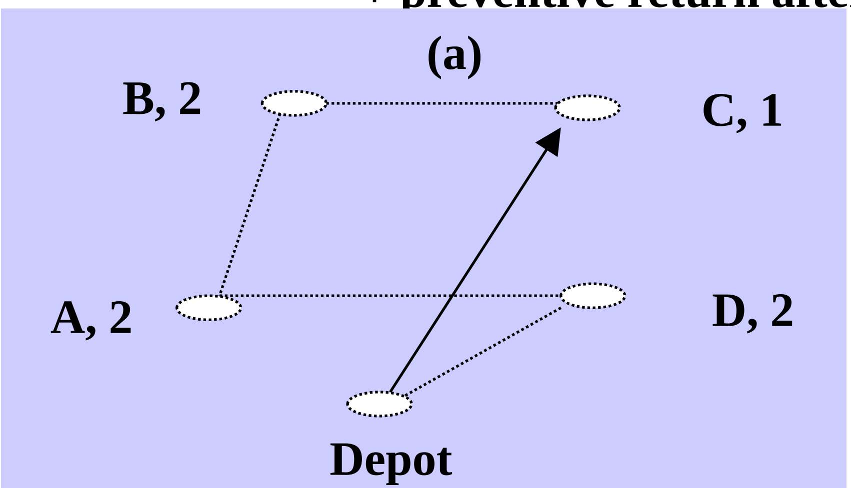
Be clever : use a recourse policy !

➔ a priori route
➔ rules for return trip/preventive returns

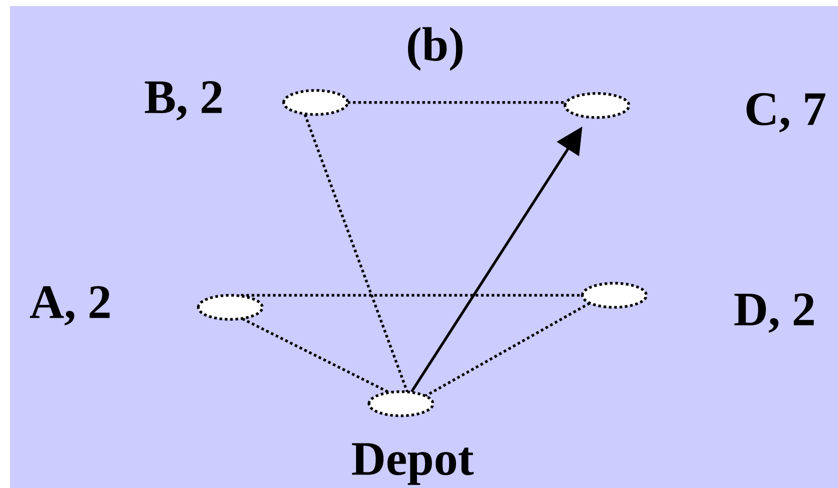
Optimal recourse policy

A priori route : C,B,A,D

+ preventive return after B when demand in C is 7



Effective length = 11
« demand in C = 1 »



Effective length = 14
« demand in C = 7 »

RP = 12.5 = expected effective length, under recourse policy

Classical relationships

$WS \leq RP \leq EEV$

EVPI = Expected Value of Perfect Information = RP - WS

VSS = Value of Stochastic solution = EEV - RP

Routing example **:** WS = 12, RP = 12.5, EEV = 14
EVPI = 0.5, VSS = 1.5

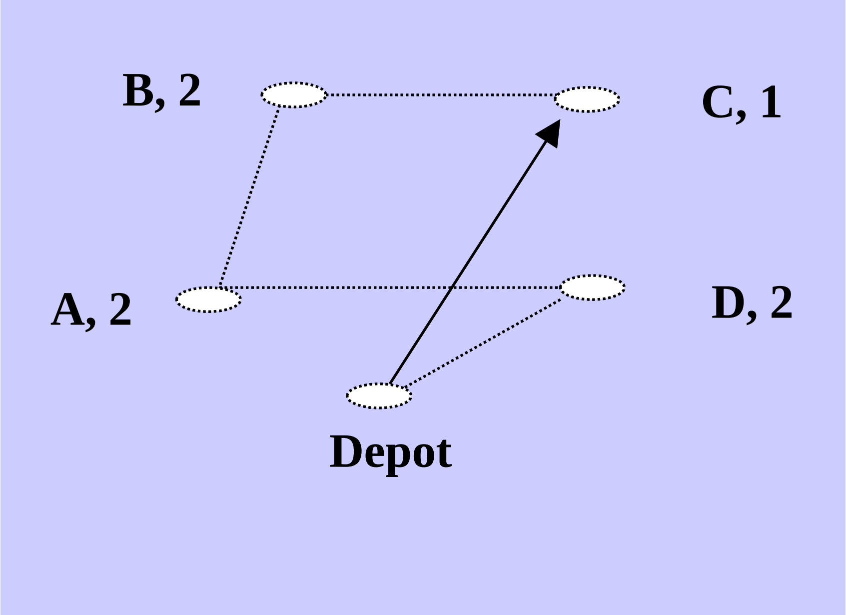
These values can only be computed **a posteriori.**
The decision of solving a stochastic program or not must be made a priori.

Principles & modelling: « Identical as in continuous stochastic programming »

Why not solving a series of deterministic programs to get a number of typical « good » solutions, and select the best one according to the expected cost ?

Answer : some solutions cannot be found by a deterministic program.

The optimal a priori solution of the stochastic routing example will never be obtained by a deterministic program



Assume any change of data (demand, vehicle capacity)
Then, when the vehicle can handle the total demand, it will always follow the shortest route (the TSP route)
If it cannot handle the total demand in one leg, it will always follow the best two legs route, not this one.

Additional example : LTL movements (Lium, Crainic, Wallace TS08)

Modelling Uncertainty : Recourse Models

Min $c.x + E_{\xi} Q(x,\xi)$

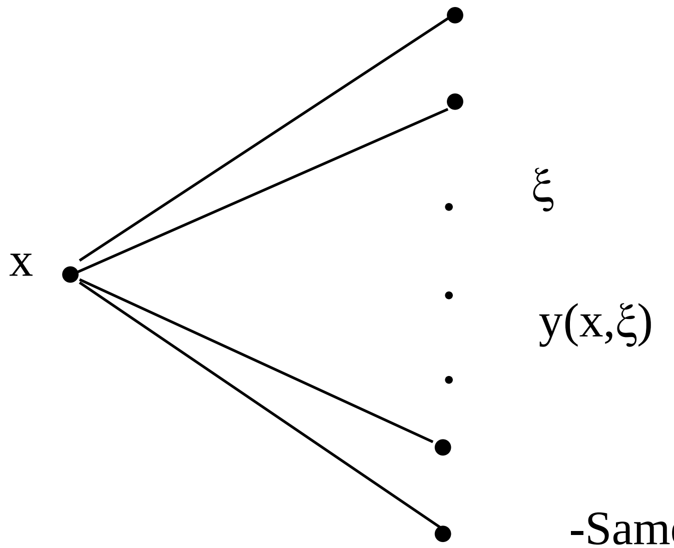
s.t. $A.x = b$, $x \in X$

where $Q(x,\xi) = \min \{q.y \mid Wy = h - Tx , y \in Y \}$

x = first-stage decisions
ξ= stochastic components of q, h, T, W
$y(x, \xi)$ = second-stage decisions

x --> ξ --> y
non anticipative or implementable

difficulty is in $Q(x,\xi)$ and
« dimension » of ξ

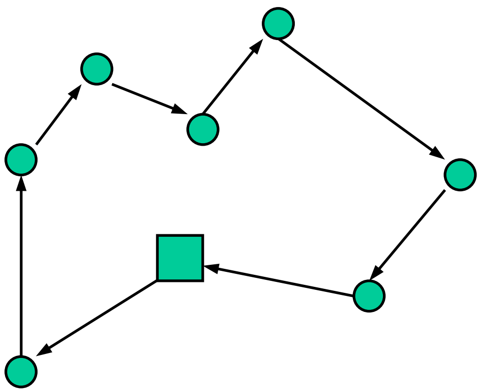


If y is continuous, $Q(x, \xi)$ is piecewise linear and convex
→ may apply L-shaped for discrete ξ

-Same representation when ξ is a continuous r.v.

-**Integer extensions**: when x and/or **y** must be integer

To be or not to be Integer : stochastic TSP



$x = (x_{ij})$, 1 if arc (i,j) is travelled, 0 otherwise
binary first-stage

a. **Random demand** & failures
$\xi = (d_i)$, the demand on i
y_i = **binary if failure occurs in i**
→ **binary (difficult) second-stage**

b. **Random travel times**
$\xi = (t_{ij})$, the travel time on arc (i,j)
T = time limit

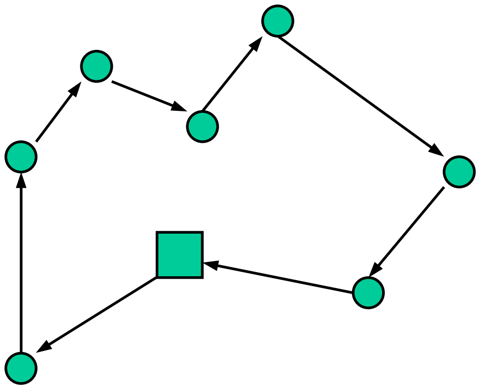
q = unit penalty for overtime
y = overtime = $y(x, \xi)$

$Q(x,\xi) = \min \{q.y \mid y \geq \Sigma_{ij}\ t_{ij}\ x_{ij} - T , y \geq 0 \}$

$= q\,(\Sigma_{ij}\ t_{ij}\ x_{ij} - T)^+$

« easy problem » as second stage is continuous

TSP with stochastic travel times : integer second-stage



Vehicles collecting money (Lambert, Laporte, Louveaux COR93)

« if vehicle arrives late, then money is value of tomorrow »

Penalty is no longer proportional to tardiness,
paid as soon as time limit is exceeded
$\Rightarrow$ **Indicator variable == binary variable**

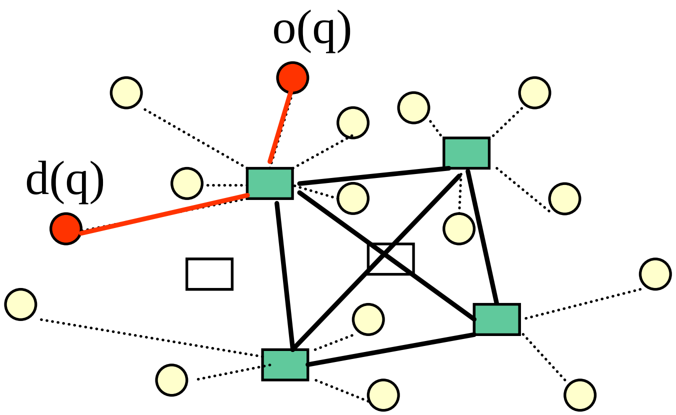
y = 1 if vehicle arrives late, 0 otherwise

$Q(x,\xi) = \min \{q.y \mid My \geq \Sigma_{ij}\ t_{ij}\ x_{ij} - T, \quad y \in \{0,1\} \}$

And much more difficult
if more than one route

Hub Location Problem

O'Kelly (TS 86, EJOR87)
Contreras, Cordeau, Laporte (EJOR11)

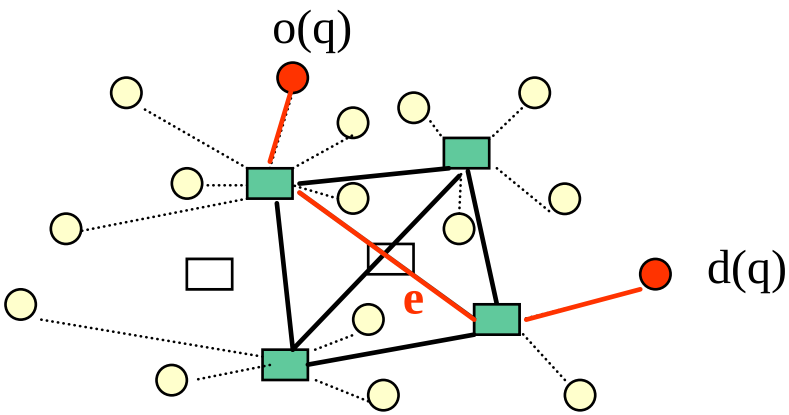


Select a set of Hub nodes

- that are fully connected
- to serve O-D demands: commodity q

- Using hubs

Hub Location Problem



□ Potential hub locations

○ Clients

Select a set of Hub nodes

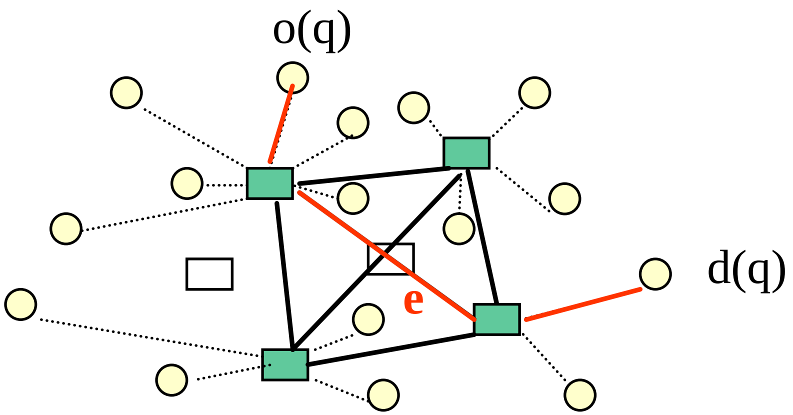
- that are fully connected
- to serve O-D demands: commodity q
  - Using hubs
  - or hub connections, using edge **e** between two hubs

Decisions

x_i = open hub i $\in$ H
y_{eq} = commodity q is served through edge **e** $\in$ E

with
f_i = fixed cost of opening hub i $\in$ H
c_{eq} = cost of serving commodity q through edge e $\in$ E

Stochastic Hub Location Problem



Uncertainty may come from

- random demands
- random costs

Decisions

x_i = open hub $i \in H$ = first-stage binary

y_{eq} = q is served through $e \in E$ = second-stage binary

Very large second-stage (already in the deterministic case = large number of q 's)

Many other examples

- Unit commitment (Takriti, Birge, Long IEEE 96)

- Production planning (lot sizing) (Haugen, Løkketangen, Woodruff EJOR01)

- Ground Holding Airlines Operations (Ball et al OR 03)

- Capacity expansion (Ahmed, Garcia AOR 03)

And we want to solve also generic problems (no specific structure)

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M. Van der Vlerk Stochastic Integer Programming Bibliography
http://mally.eco.rug.nl/index.html?biblio/sip.html

Main reference used in this talk

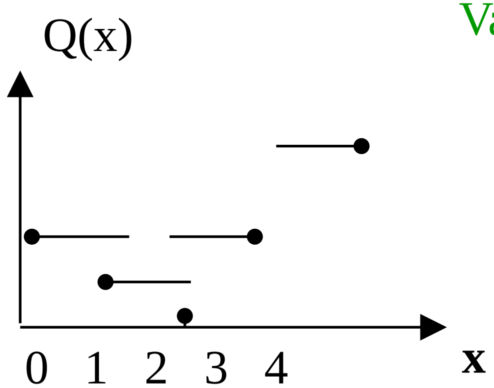
J. Birge, F. Louveaux, Introduction to Stochastic Programming, Springer 2011

Presentation outline

- **Modelling**
- **Difficulty**

Difficulty of S.I.P.

$\mathbf{Q}(x) = \min \{ 2 \cdot y_1 \cdot + y_2 \mid \cdot y_1 \geq 2 - x ,\ y_2 \geq x - 2,\ y \geq 0,\ y \text{ integer} \}$



Value of a deterministic integer program (Blair, Jeroslow MP82)

$\mathbf{Z}(b) = \min \{ q.y \mid W.y \geq b \quad y \geq 0,\ y \text{ integer} \}$

Not continuous,
Not convex,
Not…..

Subadditive : $Z(u+v) \leq Z(u) + Z(v)$
Non-decreasing
Lower semi-continuous

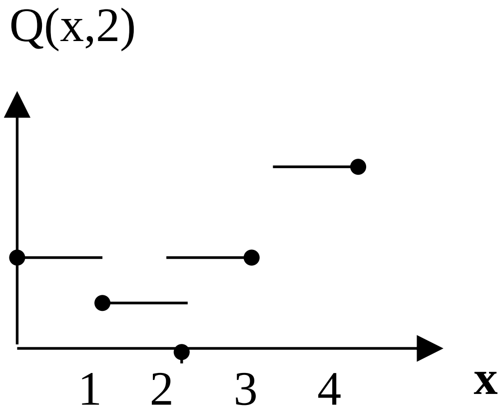
If W and b are integer, Z(b) is
piecewise constant on some multidimensionnal cells

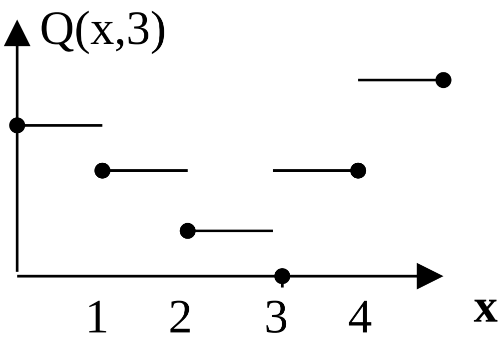
Stochastic case:
In addition, dependance on x

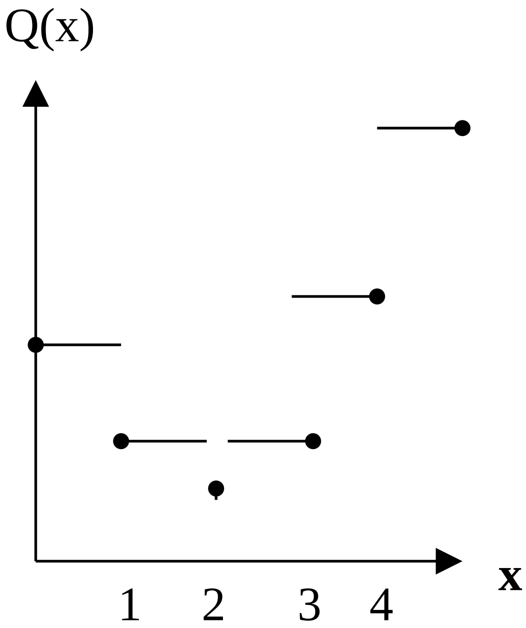
Difficulty of S.I.P. : Taking expectations

$Q(x,\xi) = \min \{ 2 \cdot y_1 + y_2 \mid \cdot y_1 \geq x - \xi , y_2 \geq \xi - x, y \geq 0, \text{integer} \}$

ξ = 2 or 3, with probability 1/2 each.







Continuous Cumulative distr. $F(\xi) = 2 - 2/\xi , \xi \in [1,2]$

Q(x) is **not convex** (but continuous)

Stougie(1985)
Schultz (MP03)

Presentation outline

- **Modelling**
- **Difficulty**
- **Exact Methods**
  - **Simple Integer**

Continuous Simple recourse

Wets, Stochastics 83

$$Q(x,\xi) = \min \{q^+ y^+ + q^- y^- \mid y^+ - y^- = \xi - Tx, \quad y^+, y^- \geq 0 \}$$

$$= \min \{q^+ y^+ + q^- y^- \mid y^+ - y^- = \xi - \chi, \quad y^+, y^- \geq 0\}$$

with $\boldsymbol{\chi} = \mathbf{T.\, x}$ (tender)

Any difference between χ and ξ is corrected by a recourse action y^+ or y^-

Simple integer recourse (van der Vlerk 95)

$$Q(x,\xi) = \min \{q^+ y^+ + q^- y^- \mid y^- \geq T.x - \xi, \quad y^+ \geq \xi - Tx, \quad y^+, y^- \geq 0, \textbf{integer} \}$$

$$= \min \{q^+ y^+ + q^- y^- \mid y^- \geq \chi - \xi, \quad y^+ \geq \xi - \chi, \quad y^+, y^- \geq 0, \text{integer} \}$$

with $\boldsymbol{\chi} = \mathbf{T.x}$ (tender)

Any difference between χ and ξ must be corrected by an **integer** recourse action

Differences can be computed componentwise

$$y_i^- = \lceil \chi_i - \xi_i \rceil^+, \quad y_i^+ = \lceil \xi_i - \chi_i \rceil^+$$

$$Q(x) = E_\xi Q(x,\xi) = E\left[\Sigma_i q_i^- \lceil \chi_i - \xi_i \rceil^+ + \Sigma_i q_i^+ \lceil \xi_i - \chi_i \rceil^+ \right], \qquad \text{with } \chi = T.x$$

All we have to understand are uni-dimensional functions of the form

$$u(x) = E_\xi \lceil \xi - x \rceil^+$$

and

$$v(x) = E_\xi \lceil x - \xi \rceil^+$$

Example : ABC airlines is offering a Tenerife-Fuerteventura flight roundtrip at 146 euros, on a ATR42 with 48 seats . They want to propose a full-fare ticket at 219 euros , allowing flexible reservations. They assume large demand for low fare & a random demand ξ for the full fare ticket.

How many seats should be reserved for the full fare (no overbooking) ?

Decision : x seats to reserve for full-fare

Remaining 48-x seats= certain revenue= 146(48-x)

Full-fare, random demand → revenue= 219 min (x, ξ)

$$\textbf{min} \; \{ -146(48-x) - E_\xi \, 219\, y, \quad y \le x, \quad y \le \xi, \quad 0 \le x \le 48, \quad x, y \in Z^+ \}$$

Use $y^+ = \xi - y$ or $y = \xi - y^+$ → $y \le x$ is $y^+ \ge \xi - x$

$y \le \xi$ is $y^+ \ge 0$

$219y = 219\,\xi - 219\, y^+$

$$-219\,\mu - 7008 + \quad \min \{146x + 219 \; E_\xi \lceil(\xi - x)\rceil^+ \}$$

$$\min \{146x + 219 \; u(x), \quad 0 \le x \le 48, \quad x \in Z^+ \}$$

Expected surplus : $u(x) = E_\xi \lceil \xi - x \rceil^+$

« Surplus » = surplus of « demand ξ » versus « production x »

$u(x) = E_\xi \lceil \xi - x \rceil^+ = \Sigma_{j \geq 1} \; j.\, P(\lceil \xi - x \rceil^+ = j)$ as $\lceil \xi - x \rceil \in Z$

$= \Sigma_{j \geq 1} \; j.\, P(j-1 < \xi - x \leq j) = \Sigma_{j \geq 1} \; j.\, P(j+x-1 < \xi \leq j+x)$

$= \Sigma_{j \geq 1} \; j.\, [F(j+x) - F(j+x-1)]$ with **F(t) = P($\xi \leq$ t) the cumulative distribution of ξ**

$= \Sigma_{j \geq 1} \; j.\, [F(j+x) -1 + 1 - F(j+x-1)]$ to have F(x)-1, a value -->0

$= -(1-F(x+1)) + 1-F(x) - 2(1-F(x+2)) + 2(1-F(x+1)) - 3(1-F(x+3)) + 3(1-F(x+2))$ ..

$= 1 - F(x) + 1-F(x+1) + 1- F(x+2) + \ldots$

$= \Sigma_{k=0}^{\infty} (1 - F(x+k))$

$u(x) = \Sigma_{k=0}^{\infty} (1 - F(x + k))$

Louveaux, van der Vlerk (MP93)

Bad news : infinite sum

Finite calculation of u(x)

- ξ has **finite range** : stop when F(.) = 1
- **Analytical expressions** exist : exponential distribution
- **Poisson** : Use u(0) and u(0) - u(n) = **first n terms** in u(x)
- There are good bounds when restricting u(x) to its **first n terms**

$v(x) = \Sigma_{k=0}^{\infty} \hat{F}(x - k)$ with $\hat{F}(t) = P(\xi < t)$

Same properties as u(x)

Use first n terms

$$u(x+n) = u(x) - \Sigma_{k=0}^{n-1} (1-F(x+k))$$

Proof

$u(x) = \Sigma_{k \geq 0} (1 - F(x + k))$
$u(x+1) = \Sigma_{k \geq 0} (1 - F(x + k+1))$

Thus, u(x) contains one extra term 1-F(x) :

$$\mathbf{u(x) - u(x+1) = 1 - F(x)}$$

Similarly

$$u(x+1) - u(x+2) = 1 - F(x+1)$$

Add the two : $u(x) - u(x+2) = (1- F(x)) + (1 - F(x+1))$

Repeat…

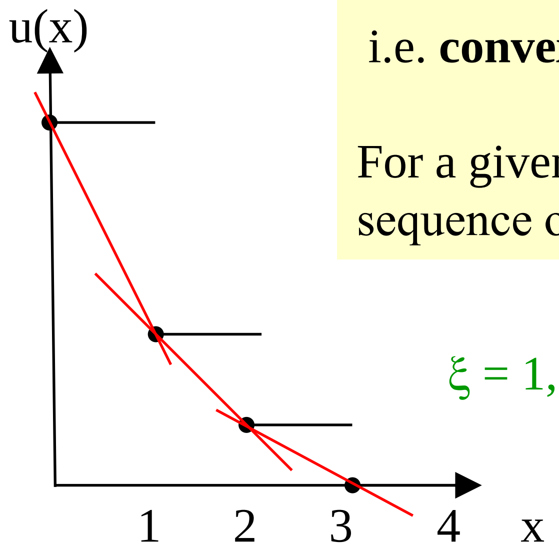
Useful when u(x) easily calculated for some x.

Example: Poisson : $u(0) = E_\xi \ulcorner \xi \urcorner^+ = E_\xi \xi = \mu$

Expected surplus : $u(x) = E_{\xi} \lceil \xi - x \rceil^{+}$
in general non convex (exception , uniform)

$u(x+1) - u(x) = 1 - F(x)$

F(x) non decreasing in x



$u(x+1) - u(x)$ is non increasing in x

i.e. **convexity property between points that are integer apart**

For a given x, the piecewice linear function that joins the sequence of points $\{x+k, u(x+k)\}$, k=0,1,… is convex

ξ = 1, 2 or 3, with probability 1/3 each.

Expected surplus : u(x) = $E_\xi \lceil \xi - x \rceil^+$
in general non convex (exception , uniform)

u(x+1) – u(x) = 1 – F(x)

F(x) non decreasing in x

u(x+1) – u(x) is non increasing in x

i.e. **convexity property between points that are integer apart**

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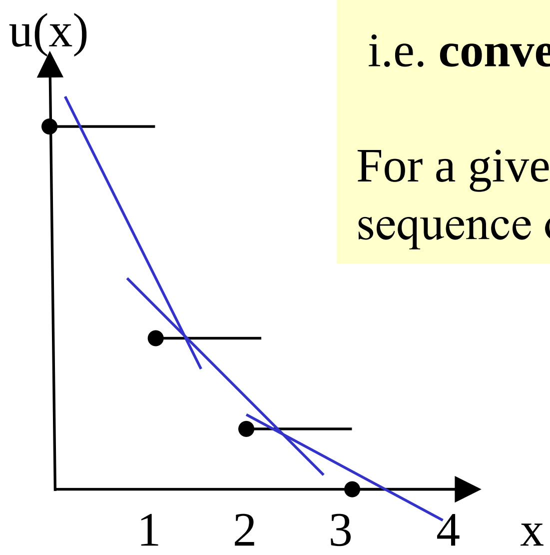
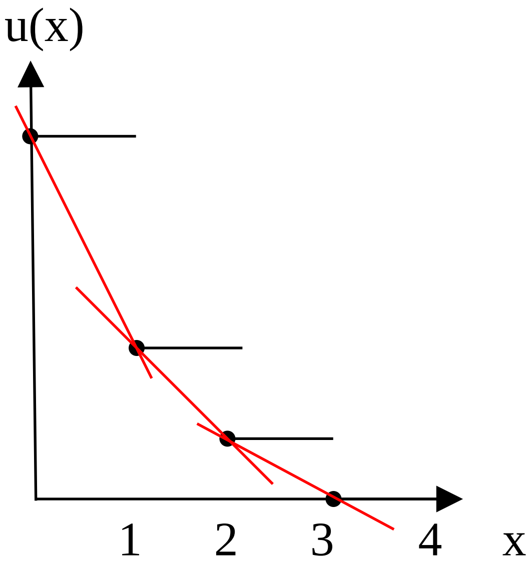


holds for any x

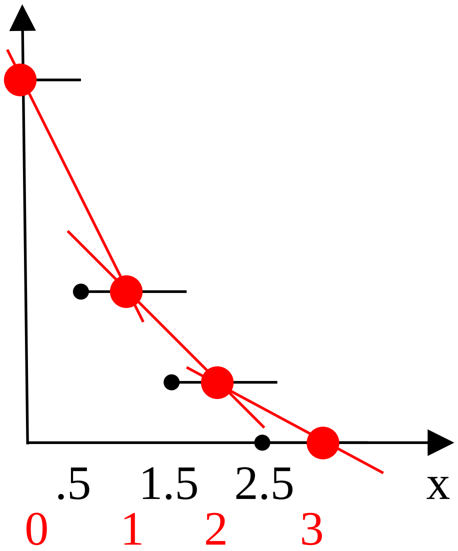
ξ = 1, 2 or 3, with probability 1/3 each.

Convexity between points that are integer apart

ξ = 1, 2 or 3, with probability 1/3 each.



ξ =0.5, 1.5, 2.5, with probability 1/3 each.



Also holds for continuous ξ

$\Rightarrow$ Exact finite method when « tenders » are integer (see next section)

Other convexifications, Van der Vlerk (MP 04)

Example : ABC airlines is offering a Tenerife-Fuerteventura flight roundtrip at 146 euros, on a ATR42 with 48 seats . They want to propose a full-fare ticket at 219 euros , allowing flexible reservations. They assume large demand for low fare & a random demand for the full fare ticket.

How many seats should be reserved for the full fare ?

$\min \{146x \;\; +219\, u(x) , \quad 0 \le x \le 48\}$

$\min \{2x \;\; +3\, u(x) , \quad 0 \le x \le 48\}$

$3\, u(0) = 9$

$2 + 3\, u(1) = 8.1494$

$4 + 3\, u(2) = 7.7470$

$6 + 3\, u(3) = 8.0163$

stop $x^*=2$

Assume demand full-fare is Poisson(3)

$u(0) == E_\xi \ulcorner \xi \urcorner^+ = E(\xi) = 3$

$u(0) = 3 \qquad u(1) = 2.0498$

$u(2) = 1.249 \qquad u(3) = 0.6721$

$u(4) = 0.3194 \qquad u(5) = 0.1346 \; \ldots..$

Presentation outline

- Modelling
- Difficulty
- Exact Methods
    - Simple Integer
    - Integer L-Shaped

Solving continuous linear recourse models (no feasibility cuts)

Min $c.x + Q(x)$
s.t. $A.x = b$
$x \in X$

Equivalent if full set of optimality cuts

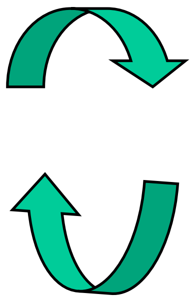
Min $c.x + \theta$
s.t. $A.x = b$
$E_s x + \theta \geq e_s$, $s = 1, \ldots S$
$x \in X$

(continuous) optimality cuts

Constructive algorithm : L-Shaped or Bender's

Proposal x^ν

(CP_ν) Min $c.x + \theta$
s.t. $A.x = b$
$E_s x + \theta \geq e_s$, $s = 1, \ldots \nu$
$x \in X$



Second-stage program $Q(x,\xi)$
$= \min \{q.y \mid Wy = h - Tx , y \in Y \}$
for all ξ

Current Problem at iteration ν
θ is a lower bound on $Q(x)$

New cut through expected dual multipliers

Integer L-shaped

Min c. x + Q(x)
s.t. A.x = b
$x \in X$

Equivalent if full set of optimality cuts

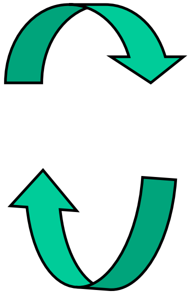
Min c. x + θ
s.t. A.x = b
$E_s\, x + \theta \geq e_s, \quad s = 1,\ldots S$
$x \in X$

optimality cuts

Constructive algorithm : Integer L-Shaped

Proposal x^{ν}

(CP_{ν}) Min c. x + θ
s.t. A.x = b
$E_s\, x + \theta \geq e_s, \quad s = 1,\ldots \nu$
$x \in X$



Second-stage program $Q(x,\xi)$
$= \min\{q.y \mid Wy = h - Tx,\ y \in Y,$ **integer**$\}$
For all ξ

Current Problem at iteration ν
θ is a lower bound on Q(x)

New cut through integrality arguments ???

Integer L-Shaped

First-stage binary variables, « any » second_stage

Finiteness comes from « finitely » many first-stage solutions,
successively eliminated by so-called « optimality cuts »

Apply a B&Cut algorithm with extended rules (as objective is « estimated »)

Conditions:

- Q(x) can be « easily » computed for a given x

- ability to obtain optimality cuts

and lower bounding functionals at fractional points (some form of lifting of cuts)

- a good lower bound on Q(x) helps

Binary Optimality Cuts (Laporte Louveaux ORL 93)

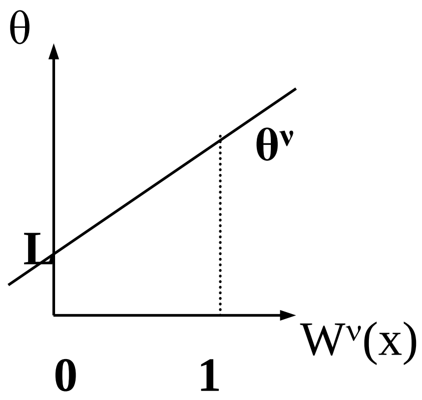
Consider a given binary solution x^ν with recourse value θ^ν

Let $S = \{i \mid x_i^\nu = 1\}$ and $S' = \{i \mid x_i^\nu = 0\}$

$$W^\nu(x) = \sum_{i \in S} x_i - \sum_{i \in S'} x_i - |S| + 1$$

$$W^\nu(x) = \begin{cases} = 1 & \text{if } x = x^\nu \\ < 1 & \text{if } x \neq x^\nu \\ \leq 0 & \text{if } x \neq x^\nu \quad x \text{ integer} \end{cases}$$

$W^\nu(x) = 1 - H(x^\nu)$, with $H(x^\nu)$ the Hamming distance to x^ν



B.O.C.1. $W^\nu(x) \leq 0$

Exclude current solution

B.O.C.2. $\theta \geq L + (\theta^\nu - L) \cdot W^\nu(x)$

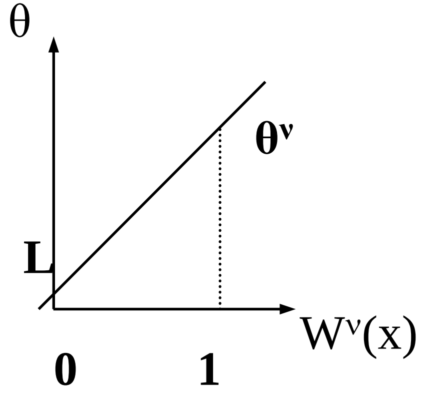
Bound the recourse function with exact value in x^ν

Example

Assume an object can be obtained from 4 sources (copper from mines, e.g.).
Let $x_i = 1$ if one invest in i, 0 otherwise.
If one invests in i, one gets a random return $\xi_i \sim P(.)$, with parameters 4,5,2,3 respectively.
In the second stage, a penalty 40 is paid if the target T=8 is not attained.

$$Q(x) = 40 * P(\Sigma_i \xi_i x_i < 8)$$

L is obtained when $x_i = 1$ for all i → $L = 40 * P(\text{Poisson}(14) < 8) = 1.265$



Consider $x^\nu = (1,0,0,1)$.

$\Sigma_i \xi_i x_i \sim P(7)$.

$\theta^\nu = 40 * 0.5987 = 23.948$

$S = \{1,4\}$ and $S' = \{2,3\}$

$W^\nu(x) = x_1 + x_4 - x_2 - x_3 - 1$

B.O.C.2. $\theta \geq 1.265 + 22.683\, W^\nu(x)$

$\theta \geq -21.418 + 22.683(x_1 + x_4 - x_2 - x_3)$

Lifting:

$W^{\nu}(x) = x_1 + x_4 - x_2 - x_3 - 1$

Consider the case when $W^{\nu}(x) = 0$. This can come from only two types of changes:

either x_2 or x_3 goes to 1 while x_1 and x_4 are unchanged,

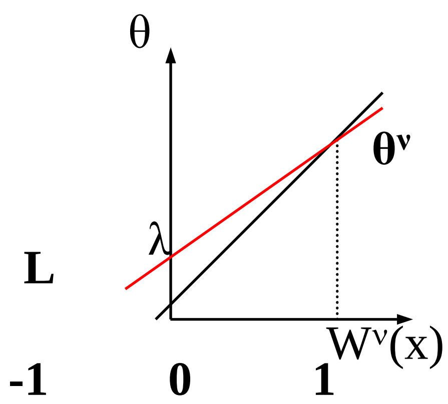
either x_1 or x_4 goes to 0 while x_2 and x_3 are unchanged

In the first case, Q(x) will decrease.

A lower bound λ on Q(.) is obtained when x_2 goes to 1 .

Then $\Sigma_i \xi_i x_i \sim P(12)$, with $P(\Sigma_i \xi_i x_i < 8) = 0.0895$ → $\boldsymbol{\lambda = 3.58}$

In the second case, Q(x) increases.



→Draw a cut through θ^{ν} and λ provided it goes below L at -1

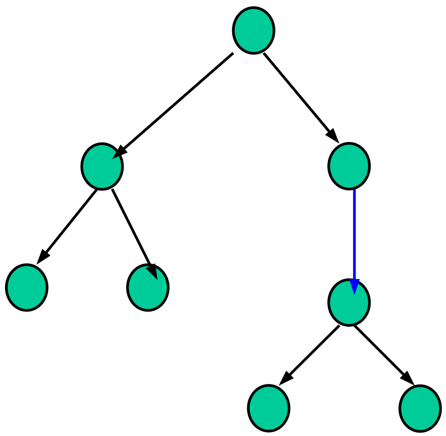
B.O.C.2.lifted	$\theta \geq 3.58 + 20.368\, W^{\nu}(x)$
$\theta \geq -16.788 + 20.368(x_1 + x_4 - x_2 - x_3)$	

Integer L-Shaped: binary first-stage

$$
\begin{aligned}
(CP_\nu) \quad & \text{Min } c.x + \theta \\
& \text{s.t. } A.x = b \\
& E_s x + \theta \geq e_s, \quad s = 1,\ldots S \\
& x \in X
\end{aligned}
$$

Current Problem at iteration ν
Includes continuous + binary O.C.
θ is a lower bound on Q(x)

(Modified) Branch &Cut scheme:



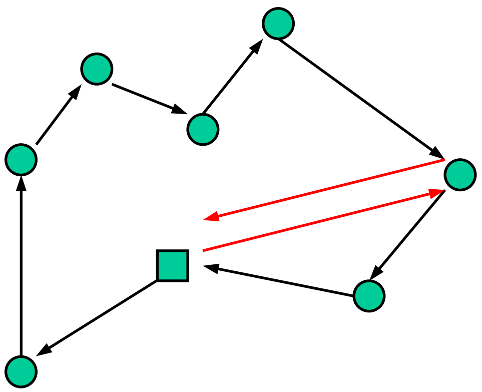
Add cuts
When needed do branching on fractional solutions
When an integer solution is found, add a B.O.C.

Nodes are eliminated only because :

- **infeasible**
- **worse solution than best known**

If possible, try to get B.O.C. at fractional first-stage solutions

VRP with stochastic demand : L & Q(x)



ξ_i = quantity to be collected in i

D = vehicle capacity

Recourse action in case of failure:

Return trip to depot

L= P(total demand > D). $\min_{i \in V} \{2c_{0i}\}$

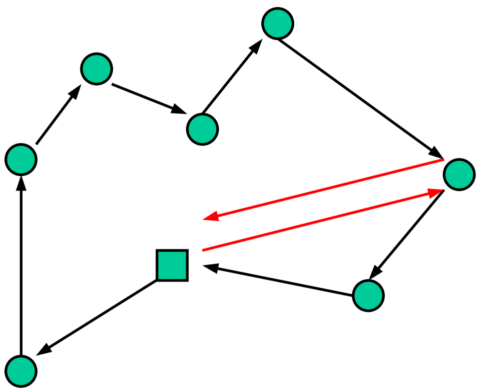
Consider a given route, say {0,1,2,…,n,0}
It has two orientations (clockwise and anticlockwise), $\lambda = 1,2$

$Q^{\lambda}(x) = \Sigma_j$ P (failure occurs at j). $2.c_{j0}$

$Q(x) = \min \{Q^1(x), Q^2(x)\}$

Assumptions :
- no preventive return,
- no exact stockout,
- no 2 failures on a route

VRP with stochastic demand : L & Q(x)



ξ_i = quantity to be collected in i
D = vehicle capacity

client
depot

Consider a given route, say {0,1,2,…,n,0} and a given orientation

Assumptions :
- no preventive return,
- no exact stockout,
- no 2 failures on a route

E_j = {cumulative demand up to j exceeds vehicle capacity}

{failure occurs at j} = $E_j \cap \overline{E}_{j-1}$ with $\overline{E}_j$:= « not E_j »

$P(E_j) = P(E_j \cap \overline{E}_{j-1}) + P(E_j \cap E_{j-1}) = P(E_j \cap \overline{E}_{j-1}) + P(E_{j-1})$

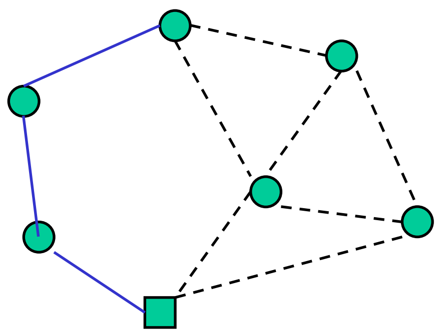
as E_{j-1} implies E_j.

→ $P(E_j \cap \overline{E}_{j-1}) = P(E_j) - P(E_{j-1}) =$

$= P(\Sigma_{1 \le s \le j}\, \xi_s > D) - P(\Sigma_{1 \le s \le j-1}\, \xi_s > D)$ complement both to 1

$= P(\Sigma_{1 \le s \le j-1}\, \xi_s \le D) - P(\Sigma_{1 \le s \le j}\, \xi_s \le D) = F_{j-1}(D) - F_j(D)$

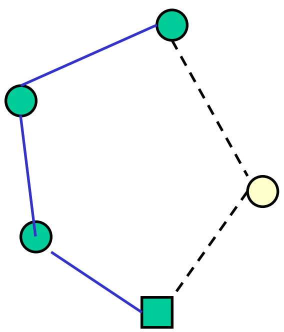
Lower Bounding functionals at fractional points (m = 1) (Hjörring - Holt AOR98)



-------- fractional arcs

Partial Route h = $(0, 1, 2, \ldots, r)$,

$H = \{1, 2, \ldots, r\}$



Dummy

demand $= \sum_{i \in V/H} d_i$

distance to depot = $\min_{i \in V/H} \{c_{0i}\}$

θ_h = expected cost of dummy route

$\Rightarrow$ θ_h := lower bound on Q(x) for any solution including h as a partial route

Cut : $\theta \geq (\theta_h - L)\ (\sum_{k<r} x_{k,k+1} - (r-1)) + L$

since $\sum_{k<r} x_{k,k+1} \geq r$ iff x contains partial route h

Extra efficiency through local branching: (Rei et al Informs J. of Computing09)

Extensions to any m (Laporte, Louveaux, Vanhamme OR 02)

Cuts based on $r \leq m$ partial routes

Advanced techniques for lower bounds

m = 3, n = 50
m = 2, n = 100

Good treatment of r.v.
Poisson, Normal

No easy second-stage formulation

Especially if more than one vehicle

Branch&Price Christiansen, Lysgaard (ORL07)

Cuts based on subsets Jabali et al (12)

Presentation outline

- **Modelling**
- **Difficulty**
- **Exact Methods**
  - **Simple Integer**
  - **Integer L-Shaped**
  - **Finiteness / Branching (in the second stage)**

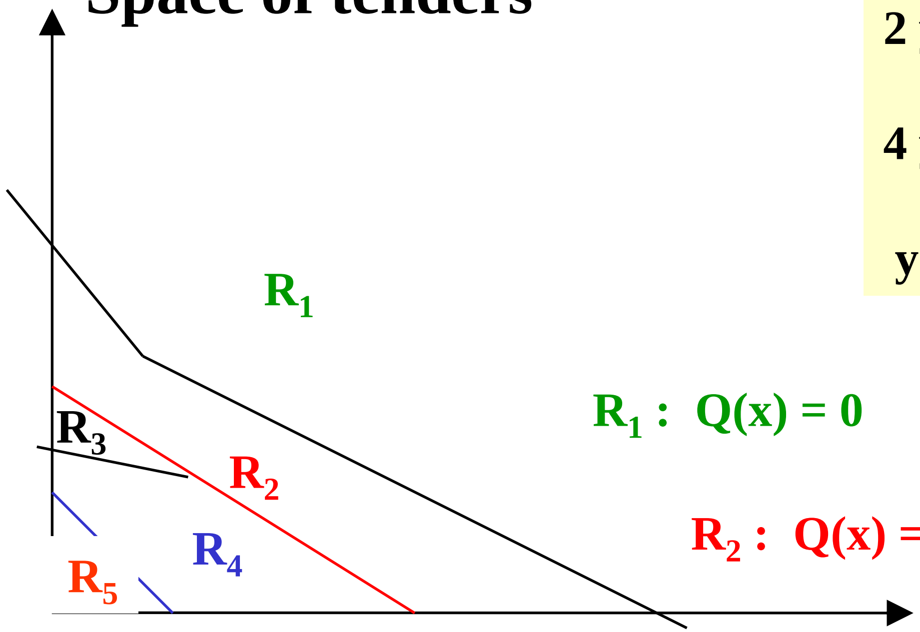
Finiteness : Space of tenders

$$Q(x) = \min\ 5y_1 + 3y_2$$

$$2y_1 + 3y_2 \geq 5 - x_1 - 2x_2$$

$$4y_1 + y_2 \geq 3 - x_1 - x_2$$

$$y_1, y_2 \geq 0, \text{ integer}$$



R_1 : $Q(x) = 0$ R_1 : $\{x \mid x_1 + 2x_2 \geq 5, x_1 + x_2 \geq 3\}$

R_2 : $Q(x) = 3$ R_2 : $\{x \mid x_1 + x_2 \geq 2\} / R_1$ as $y_1 = 0$ $y_2 = 1$ is optimal

R_3 : $Q(x) = 5$ as $y_1 = 1$ $y_2 = 0$ is optimal R_3 : $\{x \mid x_1 + 2x_2 \geq 3\} \backslash R_2 \backslash R_1$

R_4 : $Q(x) = 6$ as $y_1 = 0$ $y_2 = 2$ is optimal R_4 : $\{x \mid x_1 + x_2 \geq 1\} \backslash R_3 \backslash R_2 \backslash R_1$

R_5 : $Q(x) = 8$ as $y_1 = 1$ $y_2 = 2$ is optimal R_5 : $\{x \mid x_1 \geq 0, x_2 \geq 0\} \backslash R_4 \backslash R_3 \backslash R_2 \backslash R_1$

Transformation

$$\min cx + Q(x)$$
$$A x = b$$
$$x \geq 0 \text{ integer}$$

$$Q(x) = E_\xi\, Q(x, \xi)$$

$$Q(x,\xi) = \min\ 5 y_1 + 3y_2$$
$$2 y_1 + 3 y_2 \geq \xi_1 - x_1 - 2x_2$$
$$4 y_1 + y_2 \geq \xi_2 - x_1 - x_2$$
$$y_1, y_2 \geq 0, \text{ integer}$$

$$\min cx + \psi(\chi)$$
$$A x = b$$
$$Tx = \chi$$
$$x \geq 0 \text{ integer}, \chi \text{ integer}$$

$$\psi(\chi, \xi) = \min\ 5 y_1 + 3y_2$$
$$2 y_1 + 3 y_2 \geq \xi_1 - \chi_1$$
$$4 y_1 + y_2 \geq \xi_2 - \chi_2$$
$$y_1, y_2 \geq 0, \text{ integer}$$

$$\chi_1 = x_1 + 2x_2$$
$$\chi_2 = x_1 + x_2$$

$$\psi(\chi) = E_\xi\, \psi(\chi, \xi)$$

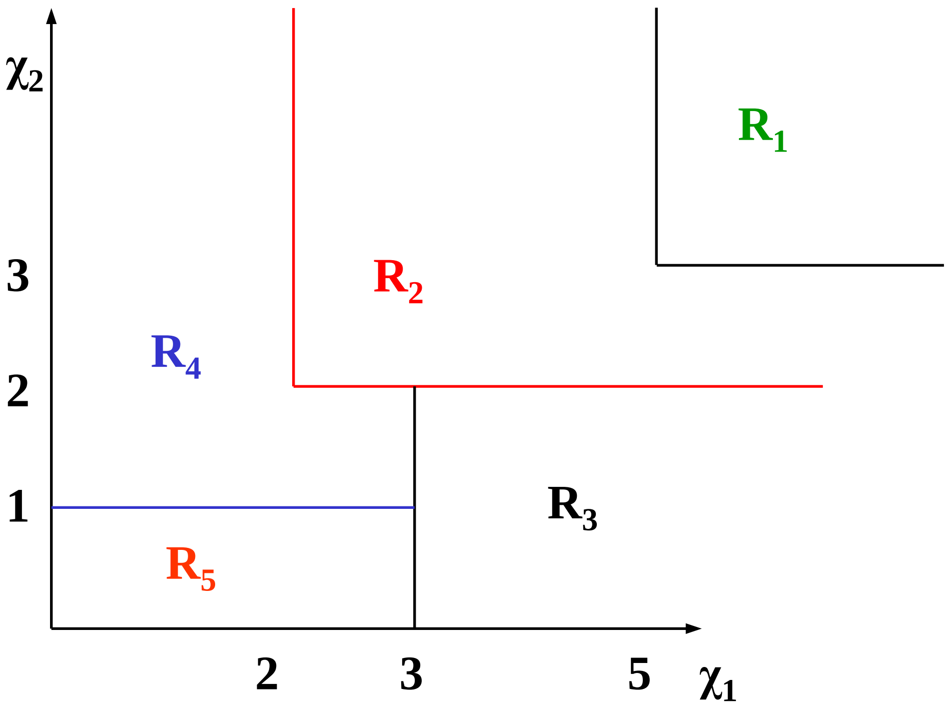
Transformation

$$\psi(\chi) = \min\ 5y_1 + 3y_2$$

$$2y_1 + 3y_2 \geq 5 - \chi_1$$

$$4y_1 + y_2 \geq 3 - \chi_2$$

$$y_1, y_2 \geq 0, \text{ integer}$$

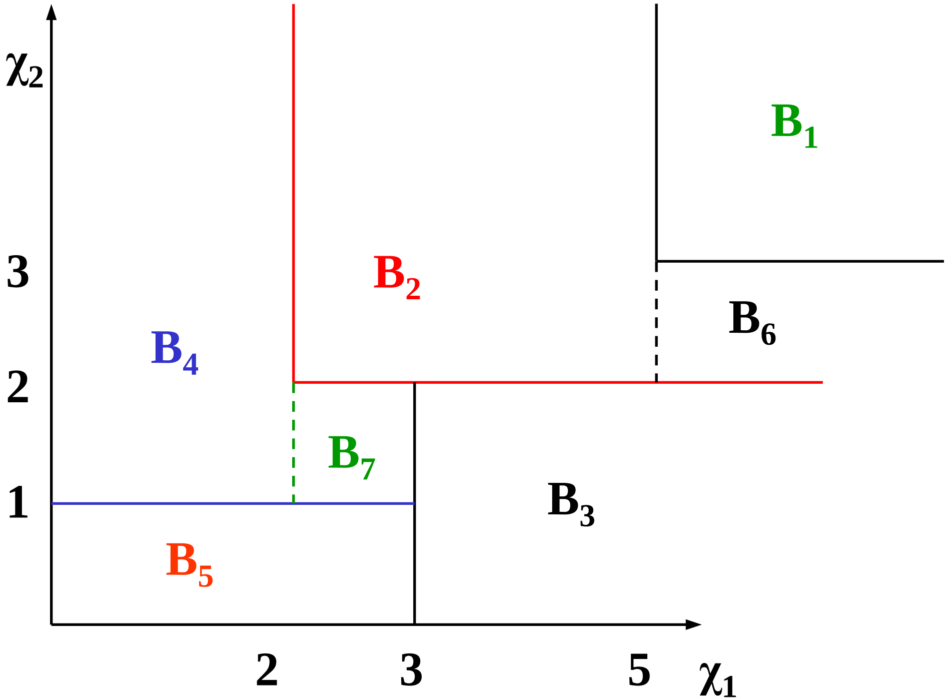


$R_1 : \psi(\chi) = 0$ $R_2 : \psi(\chi) = 3$

$R_3 : \psi(\chi) = 5$ $R_4 : \psi(\chi) = 6$ $R_5 : \psi(\chi) = 8$

Regions are not closed : e.g. R5 = $\{0 \leq \chi_1 < 3, 0 \leq \chi_2 < 1\}$ and not convex

Transformation



Hyper-rectangles or boxes of the form $\Pi_i \; [l_i , u_i - \varepsilon]$

Intersections over ξ will keep discontinuities at integer points

Branch & Bound by partitionning the (finite) space of χ

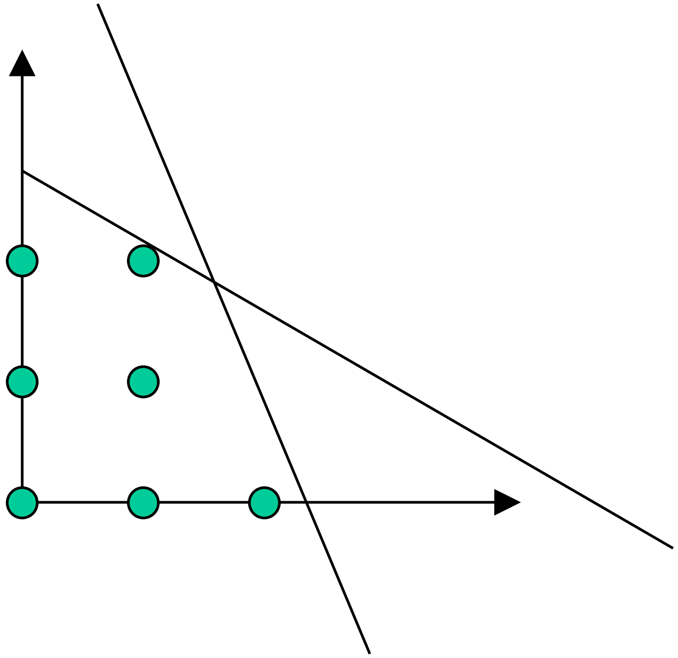
Ahmed,Tawarmalani, Sahinidis (MP 04)
Kong, Schaeffer, Hunsanker (MP 06)

Pure IP
Fixed tender
W may be random

Presentation outline

- **Modelling**
- **Difficulty**
- **Exact Methods**
  - **Simple Integer**
  - **Integer L-Shaped**
  - **Finiteness / Branching**
  - **Reformulation / Valid Inequalities**

Reformulation



Integer point

LP-relaxation:

$$12\,x_1 + 5\,x_2 \le 28$$

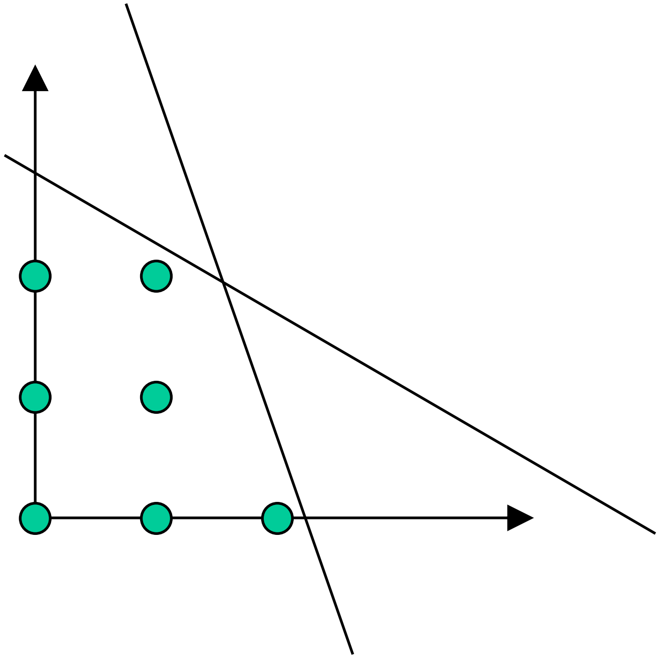
$$5\,x_1 + 12x_2 \le 30$$

$$x_1, x_2 \ge 0$$

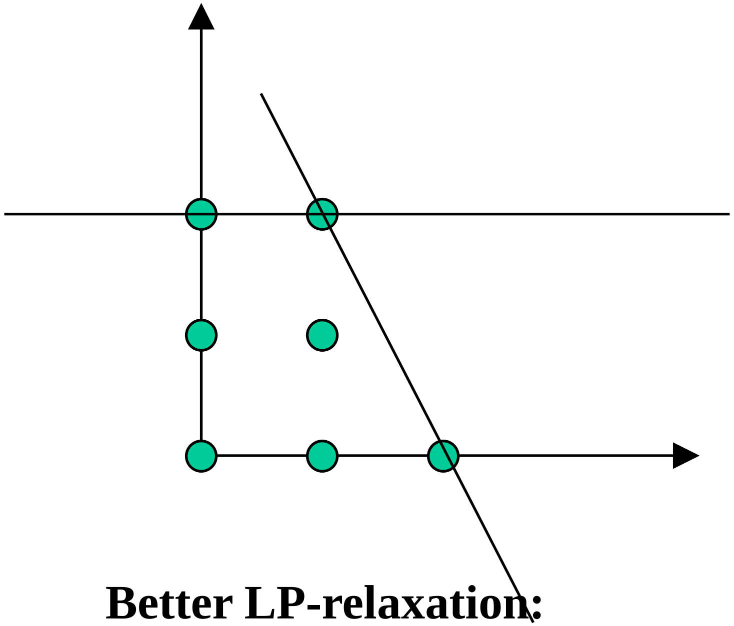
Typical LP-solution: $x_1 = 1.849 \quad x_2 = 1.563$

Fractional solution ; lots of branching

Reformulation



Typical LP-solution: $x_1 = 1 \quad x_2 = 2$



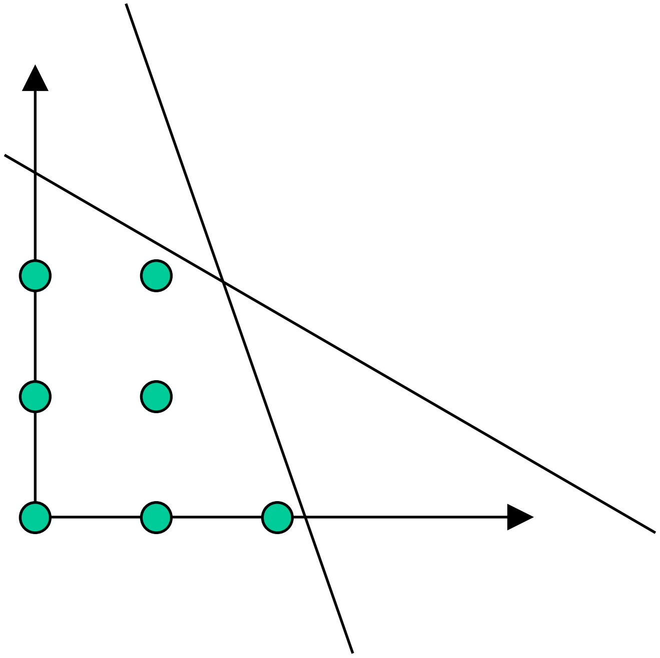
Better LP-relaxation:

$$2x_1 + x_2 \leq 4$$

$$x_2 \leq 2$$

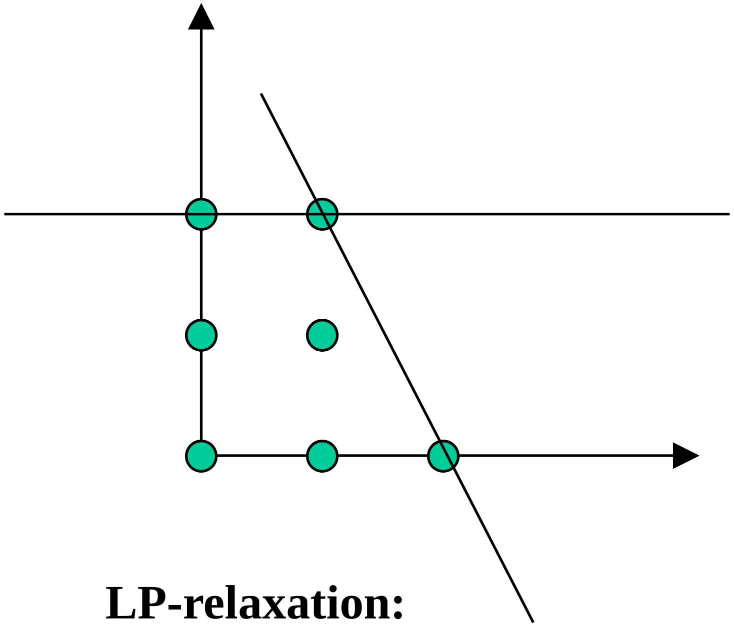
$$x_1, x_2 \geq 0$$

Reformulation



All extreme points are integer

Formulation ≡ Integer Hull



LP-relaxation:

$$2\,x_1 + x_2 \leq 4$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Reformulation/Valid Inequalities

Extensive research over the years
How to find them (separation problem)
How to use them (send groups of global constraints)
Generic or specific

$4\,y_1 + 5\,y_2 + 3\,y_3 + 6\,y_4 \le 8$
y binaries

$y_1 + y_2 \le 1$

$y_1 + y_4 \le 1$

$y_2 + y_4 \le 1$

$y_3 + y_4 \le 1$

Current solution (associate LP)
$y_1 = 1,\; y_2 = 0.8,\; y_3 = y_4 = 0$

Cover inequalities, LGCI (Gu, Nemhauser,Saveslbergh 96)

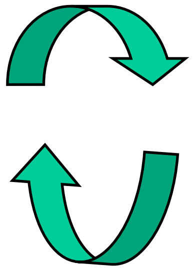
Flow cover inequalities (Q.Louveaux, L.Wolsey 05)

Gomory cuts, MIR, disjunctive cuts… (Gomory 58, Nemhauser Wolsey 90, Jeroslow 72, Balas 75)

Solving SIP

Proposal x^{ν}

(CP_{ν}) Min $c.x + \theta$
s.t. $A.x = b$
$E_s\, x + \theta \geq e_s$, $s = 1,\ldots\nu$
$x \in X$



Second-stage program $Q(x,\xi)$
$= \min \{q.y \mid Wy = h - Tx ,\ y \in Y \}$
For all ξ

Cut through expected dual multipliers

SIP: reformulate second-stage
→ polyhedral representation of second-stage

« Expected » advantages:

Use known techniques for reformulation of second-stage
Use known techniques for stochastic Bender's

Theoretical foundations : Carøe Tind (MP 98)

Disjunctive cuts (in general)

Create the **disjunction** $P=(P^0 \cup \mathbf{P}^1)$ with $P^0 = \{x \mid A^0 x \geq b^0, x \geq 0\}$ & $P^1 = \{x \mid A^1 x \geq b^1, x \geq 0\}$

Any non negative combination of the constraints is a valid inequality.

Pick some vector $u^0 \geq 0$ as combination of $A^0 x \geq b^0$; $u^0 A^0 x \geq u^0 b^0$ is valid for P^0

Similarly pick $u^1 \geq 0$ as combination of $A^1 x \geq b^1$

Then $\pi x \geq \rho$ is valid with

$\pi \geq \max \{u^0 A^0, u^1 A^1\}$

$\rho \leq \min \{u^0 b^0, u^1 b^1\}$

Indeed , if $x \in (P^0 \cup P^1)$, it must belong to one of the sets.
Say, it belongs to P^0 , then $\pi x \geq u^0 A^0 x \geq u^0 b^0 \geq \rho$
Same for P^1

To find a violated inequality at the current x^ν , solve **max violation**:

Max $\rho - \pi x^\nu$
$\pi \geq u^0 A^0$, $\pi \geq u^1 A^1$, $\rho \leq u^0 b^0$, $\rho \leq u^1 b^1$ + some normalisation , $u \geq 0$

Disjunctive cuts (for branching)

Create the **disjunction** $Y = Y^0 \cup Y^1$,
with $Y^0 = \{y \mid Wy \geq b,\ y \leq e,\ \mathbf{y_j \leq 0}\}$ & $Y^1 = Y \cap \{y \mid Wy \geq b,\ y \leq e,\ \mathbf{y_j \geq 1}\}$
with e the unit vector.

Let u^0 the multipliers of $Wy \geq b$, v^0 the multipliers of $y \leq e$ and w^0 the multiplier of $\mathbf{y_j \leq 0}$ in Y^0
Let u^1 the multipliers of $Wy \geq b$, v^1 the multipliers of $y \leq e$ and w^1 the multiplier of $\mathbf{y_j \geq 1}$ in Y^1

Then $\pi y \geq \rho$ is valid with

$$\pi \geq u^0 W - v^0 - w^0 e_j$$
$$\pi \geq u^1 W - v^1 + w^1 e_j$$

$$\rho \leq u^0 b - e v^0$$
$$\rho \leq u^1 b - e v^1 + w^1$$

To find a violated inequality at the current y^ν, solve max violation

Max $\rho - \pi y^\nu$
s.t. the above constraints and normalisation $-1 \leq \rho \leq 1$ and $-e \leq \pi \leq e$

Reformulation in Stochastic Integer Programming

- Add cuts (valid inequalities) to the second stage
- Try to create cuts that are shared by several (all) realisations ξ^k
- Warm start

Sen & Higle (MP 05)

- Use Lift&Project to generate second-stage cuts
  (Balas,Ceria, Cornuéjols MP93, Balas Perregaard MP03)
- Cuts are obtained from the solution of an LP, as a linear combination of the constraints :
  if W is fixed, **l.h.s. are independent of the realizations of the r.v.**
- Warm start as one cut is added to current basis

D1. **Many realizations of ξ**

Max $\rho - \pi y^{\nu}$

normalisation $-1 \le \rho \le 1$ and $-e \le \pi \le e$

$\pi \ge u^0 W - v^0 - w^0 e_j$

$\pi \ge u^1 W - v^1 + w^1 e_j$

$\rho \le u^0 b - e v^0$

$\rho \le u^1 b - e v^1 + w^1$

Solve one such problem for each ξ^k

Observation: constraints on π are the same when W is fixed

Alternative : one inequality $\pi y \ge \rho^k$, for each k, but same π for everybody

C^3 = **Common Cut Coefficients → common π**

C^3 Max $\Sigma_k p_k (\rho^k - \pi y^{\nu k})$

$\rho^k \le u^0 b^k - e v^0$

$\rho^k \le u^1 b^k - e v^1 + w^1$

same constraints on π

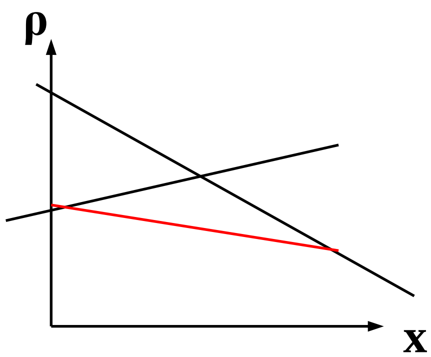
same normalisation

with $y^{\nu k}$ = solution of current second-stage for ξ^k

D2. **Cuts are dependent on x**

$\rho^k \leq u^0 b^k - e\, v^0$

$\rho^k \leq u^1 b^k - e\, v^1 + w^1$



In S.I.P., $b^k = h^k - T^k x$ is a function of x $\Rightarrow$

ρ^k is a function of x

$\rho^k \leq u^0 (h^k - T^k x) - e\, v^0 = u^0 h^k - e\, v^0 - u^0 T^k x = \alpha^0 - \beta^0 x$

$\rho^k \leq u^1 (h^k - T^k x) - e\, v^1 + w^1 = \ldots = \alpha^1 - \beta^1 x$

Solve RHS ^{k} : LP to obtain **cut valid $\forall$ x**

RHS^k = **disjunction** $(P^0 \cup \mathbf{P}^1)$

$P^0 = \{x \mid Ax \geq b, \gamma \geq \alpha^0 - \beta^0 x, x \geq 0\}$ &

$P^1 = \{x \mid Ax \geq b, \gamma \geq \alpha^1 - \beta^1 x, x \geq 0\}$

where γ represents the minimum of the two expressions and $Ax \geq b$ bounds the region where convexification occurs.

Ntaimo & Sen (JGO 04) Computational experiments

Sherali & Sen (MP 05) Add Branching in second-stage

Gade, Kuçukyavuz,Sen (MP 12) Gomory cuts when x is binary.

Presentation outline

- **Modelling**
- **Difficulty**
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  - **Simple Integer**
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  - **Reformulation / Valid Inequalities**
- **Sampling**

Sample Average Approximation Method

$\mathbf{z^* = \min \{ c.x + Q(x) \mid x \in X \}}$

with $Q(x) = E_\xi Q(x,\xi)$, and $Q(x,\xi)$ = recourse for one realization of the random variable ξ

Sampling and solution step :

Take a sample of size N, say $\xi^1, \xi^2, \xi^3, \ldots, \xi^N$ and solve

(SAA) $z_N = \min \{ c.x + 1/N \, \Sigma_{k=1,\ldots,N} Q(x,\xi^k) \mid x \in X \}$

Denote by x_N an optimal solution to (SAA)

Repeat M times the sampling and solution step

Generate M values $z_N^1, z_N^2, \ldots, z_N^M$ and
M candidate solutions $x_N^1, x_N^2, \ldots, x_N^M$

-

Sample Average Approximation Method

⇒The mean value $z_L = 1/M\ \Sigma_{i=1,..,M} z_N^i$ is, in expectation, a lower bound on z^*

$E(z_L) \leq z^*$

(Norkin, Pflug, Ruszczyński MP98, Mak, Morton,Wood ORL99)

How to choose amongst the M candidate solutions ?

Draw a new & independent sample of size S (>>N)

Select the candidate solution that does best with estimated objective function

$z_S(x) = c.\ x + 1/S\ \Sigma_{k=1,\dots,S} Q(x,\xi^k)$

Denote by x^S such a candidate solution with least $z_S(x)$ value.

$x^S \in \arg\min \{ z_S(x) \mid x \in \{x_N^1, x_N^2, \dots, x_N^M\} \}$

Sample Average Approximation Method

⇒ $z_S(x^S)$ is an unbiased estimator of $z(x^S)$ and therefore, in expectation, an upper bound on the optimal value

$$E(z_S(x^S)) \geq z^*$$

⇒ At the end, the SAA method provides

- estimators of Lower & Upper bound on z*

- an estimation of their variances &

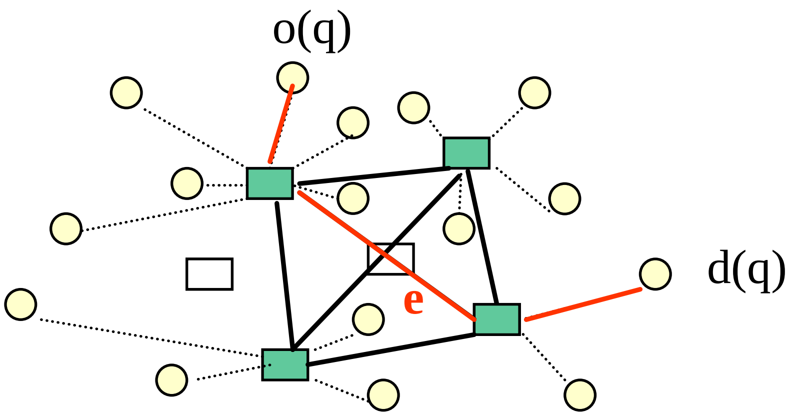
- a candidate solution with the smallest estimate objective value

Verweij et al (COA 03) …. Schütz, Tomasgard,Ahmed (EJOR 09)

SAA for chance constraint : Calafiore, Campi (M.P.05)
Nemirovski, Shapiro (SIAM J. O. 06)

Stochastic Hub Location Problem Contreras,

Contreras,Cordeau,Laporte (EJOR11)



Decisions

x_i = open hub i $\in$ H = binary

y_{eq} = q is served through e $\in$ E = binary

If only the levels of demands are random, demands can be replaced by their expected values (same route followed if uncapacitated)

Same is true for dependent random costs

Interesting/ Difficult cases = capacitated hub location with random demands
independent random costs

Solvable size

- deterministic: 500 nodes, 250,000 commodities (o(q), d(q))
- Stochastic costs with SAA +Benders with Pareto-optimal cuts :
  50 nodes 2500 commodities 1000 scenarios

Remark

- **Several additional methods are available**

- Second-stage decomposition with separable recourse

- Dual decomposition (scenario decomposition using Lagrangean relaxation w.r.t. non anticipativity constraints) Caroe Schultz (ORL99)

- Other enumerative approaches

- Stochastic B&B using statistical estimates : Norkin, Ermoliev, Ruszczyński (OR 98)

- Cuts for specific problems: e. g. lot sizing Guan, Ahmed, Nemhauser,Miller (M.P. 06)

Conclusion

Good news about S.I.P.

- **Some efficient methods are now available** — **For the users**

- **Several open questions remain** — **For the researchers**

Thank you…..