Formulations of Stochastic Programming Problems and Risk Aversion

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Uncertain Outcomes and Risk

Why Probabilistic Models?

- Wealth of results of probability theory
- Connection to real data via statistics
- Universal language (engineering, economics, medicine, . . .)

- Probability space $(\Omega, \mathcal{F}, P)$
- Decision space $\mathcal{X}$
- Random outcome (*e.g.*, cost) $Z_x(\omega)$, $\quad Z : \mathcal{X} \times \Omega \to \mathbb{R}$

Expected Value Model

$$\min_x \; \mathbb{E}[Z_x] = \int_\Omega Z_x(\omega)\, P(d\omega)$$

It optimizes the outcome on average (Law of Large Numbers?)

What is Risk?

Existence of unlikely and undesirable outcomes - high $Z_x(\omega)$ for some ω

Classical Utility Models

Expected Utility Models (von Neumann and Morgenstern, 1944)

$$\min_{x \in X} \mathbb{E}\big[u(Z_x)\big] \qquad \left(= \int_{\Omega} u\big(Z_x(\omega)\big)\, dP(\omega) \right)$$

$u : \mathbb{R} \to \mathbb{R}$ is a nondecreasing disutility function

Rank Dependent Utility (Distortion) Models (Quiggin, 1982; Yaari, 1987)

$$\min_{x \in X} \int_0^1 F_{Z_x}^{-1}(p)\, dw(p) \qquad F_{Z_x}^{-1}(\cdot) \text{ - quantile function}$$

$w : [0,1] \to \mathbb{R}$ is a nondecreasing rank dependent utility function

Existence of utility functions is derived from systems of axioms, but in practice they are difficult to elicit

Axioms of Expected Utility Theory

W is a lottery of Z and V with probabilities $\alpha \in (0,1)$ and $(1-\alpha)$, if the probability measure μ_W induced by W on $\mathbb{R}$ is the corresponding convex combination of the probability measures μ_Z and μ_V of Z and V:

$$\mu_W = \alpha\mu_Z + (1-\alpha)\mu_V.$$

We write the lottery symbolically as

$$W = \alpha Z \oplus (1-\alpha)V.$$

For law invariant preferences on the space of random vectors with values in $\mathbb{R}$, von Neumann introduced the axioms:

Independence Axiom: For all $Z, V, W \in \mathcal{Z}$ one has

$$Z \lhd V \implies \alpha Z \oplus (1-\alpha)W \lhd \alpha V \oplus (1-\alpha)W, \quad \forall\, \alpha \in (0,1)$$

Archimedean Axiom: If $Z \lhd V \lhd W$, then $\alpha, \beta \in (0,1)$ exist such that

$$\alpha Z \oplus (1-\alpha)W \lhd V \lhd \beta Z \oplus (1-\beta)W$$

Expected Utility

Integral Representation

Suppose the total preorder $\trianglelefteq$ on $\mathcal{Z}$ is law invariant, and satisfies the independence and Archimedean axioms. Then it has an "affine" numerical representation $U : \mathcal{Z} \to \mathbb{R}$:

$$U(\alpha Z \oplus (1-\alpha)V) = \alpha U(Z) + (1-\alpha)U(V).$$

If $\trianglelefteq$ is weakly continuous, then a continuous and bounded function $u : \mathbb{R} \to \mathbb{R}$ exists, such that

$$U(Z) = \mathbb{E}\big[u(Z)\big] = \int_{\Omega} u\big(Z(\omega)\big)\, P(d\omega).$$

New proof by separation theorem - D. & R. 2012

In a more general setting, we may consider only r.v. with finite moments, and then the boundedness condition on $u(\cdot)$ can be relaxed.

Risk-Averse Utility

$$U(Z) = \mathbb{E}\big[u(Z)\big] = \int_\Omega u\big(Z(\omega)\big)\, P(d\omega)$$

Monotonicity

The total preorder $\trianglelefteq$ is monotonic with respect to the partial order $\leq$, if $Z \leq V \implies Z \trianglelefteq V$.

We focus on $\mathcal{Z}$ containing integrable random vectors.

Risk Aversion

A preference relation $\trianglelefteq$ on $\mathcal{Z}$ is *risk-averse*, if $\mathbb{E}[Z|\mathcal{G}] \trianglelefteq Z$, for every $Z \in \mathcal{Z}$ and every σ-subalgebra $\mathcal{G}$ of $\mathcal{F}$.

Nondecreasing Convex Disutility

Suppose a total preorder $\trianglelefteq$ on $\mathcal{Z}$ is weakly continuous, monotonic, risk-averse, and satisfies the independence axiom. Then the utility function $u : \mathbb{R} \to \mathbb{R}$ is nondecreasing and convex.

Axioms of Dual Utility Theory (Yaari 1987)

Real random variables Z_i, $i = 1, \ldots, n$, are comonotonic, if

$$\Big(Z_i(\omega) - Z_i(\omega')\Big)\Big(Z_j(\omega) - Z_j(\omega')\Big) \geq 0$$

for all $\omega, \omega' \in \Omega$ and all $i, j = 1, \ldots, n$.

Dual Independence Axiom: For all comonotonic random variables Z, V, and W in $\mathcal{Z}$ one has

$$Z \lhd V \implies \alpha Z + (1-\alpha)W \lhd \alpha V + (1-\alpha)W, \quad \forall\, \alpha \in (0,1)$$

Dual Archimedean Axiom: For all comonotonic random variables Z, V, and W in $\mathcal{Z}$, satisfying the relations

$$Z \lhd V \lhd W,$$

there exist $\alpha, \beta \in (0,1)$ such that

$$\alpha Z + (1-\alpha)W \lhd V \lhd \beta Z + (1-\beta)W$$

Dual Utility

Affine Representation

If the total preorder $\trianglelefteq$ on $\mathcal{Z}$ is law invariant, and satisfies the dual independence and Archimedean axioms, then a numerical representation $U : \mathcal{Z} \to \mathbb{R}$ of $\trianglelefteq$ exists, which satisfies for all comonotonic $Z, V \in \mathcal{Z}$ and all $\alpha, \beta \in \mathbb{R}_+$ the equation

$$U(\alpha Z + \beta V) = \alpha U(Z) + \beta U(V).$$

Integral Representation

Suppose $\mathcal{Z}$ is the set of bounded random variables. If, additionally, $\trianglelefteq$ is continuous in $\mathcal{L}_1$ and monotonic, then a bounded, nondecreasing, and continuous function $w : [0,1] \to \mathbb{R}_+$ exists, such that

$$U(Z) = \int_0^1 F_Z^{-1}(p)\, dw(p), \quad Z \in \mathcal{Z}.$$

Proof by separation - D. & R. 2012

Risk Averse Dual Utility

$$U(Z) = \int_0^1 F_Z^{-1}(p)\, dw(p), \quad Z \in \mathcal{Z} \qquad (*)$$

Risk Aversion

A preference relation $\trianglelefteq$ on $\mathcal{Z}$ is *risk-averse*, if $\mathbb{E}[Z|\mathcal{G}] \trianglelefteq Z$, for every $Z \in \mathcal{Z}$ and every σ-subalgebra $\mathcal{G}$ of $\mathcal{F}$.

Convex Rank-Dependent Utility

Suppose a total preorder $\trianglelefteq$ on $\mathcal{Z}$ is continuous, monotonic, and satisfies the dual independence axiom. Then it is risk-averse if and only if it has the integral representation (*) with a nondecreasing and convex function $w : [0,1] \to [0,1]$ such that $w(0) = 0$ and $w(1) = 1$.

Two Objectives

- Minimize the expected outcome, the mean $\mathbb{E}[Z_x]$
- Minimize a scalar measure of uncertainty of Z_x, the risk $r[Z_x]$

$$r[Z] = \mathbb{V}\mathrm{ar}[Z] \qquad \text{(Markowitz' model)}$$

$$\sigma_p^+[Z] = \left(\mathbb{E}[(Z - \mathbb{E}Z)_+^p]\right)^{1/p} \qquad \text{(semideviation)}$$

$$\delta_\alpha^+[Z] = \min_\eta \mathbb{E}\left[\max\left(\eta - Z, \frac{\alpha}{1-\alpha}(Z-\eta)\right)\right] \qquad \text{(deviation from quantile)}$$

$r\Big[Z_x\Big]$ is nonlinear w.r.t. probability and possibly nonconvex in x

Example: Portfolio Optimization

$R_1, R_2, \ldots, R_n$ - random return rates of securities
$x_1, x_2, \ldots, x_n$ - fractions of the capital invested in the securities

Return rate of the portfolio (negative of)

$$Z_x = -\big(R_1 x_1 + R_2 x_2 + \cdots + R_n x_n\big)$$

Risk Optimization with Fixed Mean

$$\begin{aligned} \min_x \quad & r\big[Z_x\big] \\ \text{s.t.} \quad & \mathbb{E}\big[Z_x\big] = \mu \qquad \text{(parameter)} \\ & x \in X_0. \end{aligned}$$

Combined Mean–Risk Optimization

$$\min_{x \in X_0} \rho\big[Z_x\big] = \mathbb{E}\big[Z_x\big] + \kappa\, r\big[Z_x\big], \qquad 0 \le \kappa \le \kappa_{\max}$$

Interesting applications of parametric optimization

Nonlinear Programming Formulations for Discrete Distributions

Suppose Z has finitely many realizations $z_1, z_2, \ldots, z_S$ with probabilities $p_1, p_2, \ldots, p_S$

$$\rho(Z) = \mathbb{E}[Z] + \kappa\sigma_m^+[Z] = \mathbb{E}[Z] + \kappa\Big(\mathbb{E}[(Z - \mathbb{E}Z)_+^m]\Big)^{1/m}$$
$$= \textstyle\sum_{s=1}^S p_s z_s + \kappa\left(\sum_{s=1}^S p_s\Big(z_s - \sum_{j=1}^S p_j z_j\Big)_+^m\right)^{1/m}$$

Equivalent Problem (for $m = 1$ - linear programming)

$$\rho(Z) = \min_{v,\mu} \quad \mu + \kappa\left(\sum_{s=1}^S p_s v_s^m\right)^{1/m}$$
$$\text{s.t.} \quad \mu = \sum_{s=1}^S p_s z_s$$
$$v_s \geq z_s - \mu, \quad s = 1, \ldots, S$$
$$v_s \geq 0, \quad s = 1, \ldots, S$$

Application to Portfolios

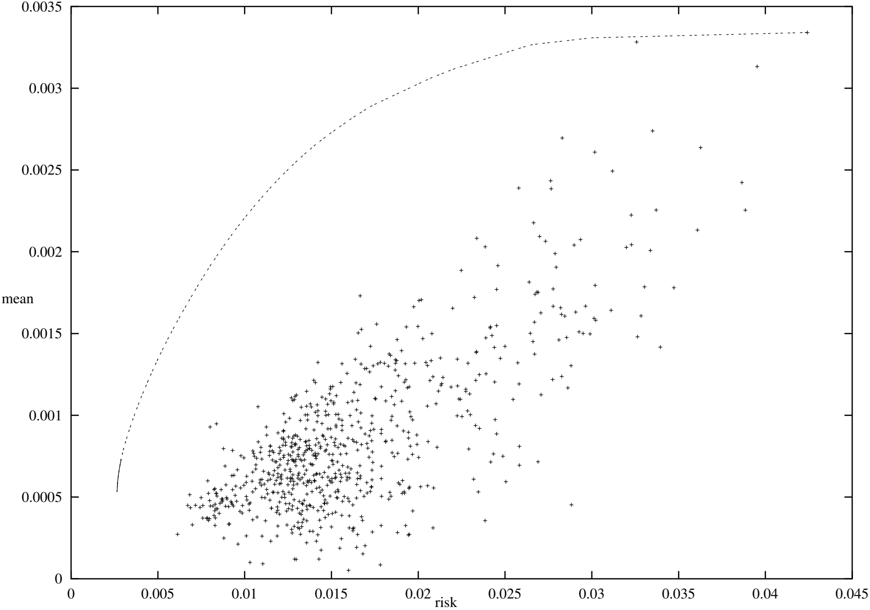
Suppose the vector of return rates has S realizations with probabilities $p_1, p_2, \ldots, p_S$

R_{js} - return rate of asset $j = 1, \ldots, n$ in scenario $s = 1, \ldots, S$

Equivalent Problem (for $m = 1$ - linear programming)

$$
\begin{aligned}
\min_{x,z,v,\mu} \quad & \mu + \kappa \Bigg(\sum_{s=1}^{S} v_s^m \Bigg)^{1/m} \\
\text{s.t.} \quad & \mu = \sum_{s=1}^{S} p_s z_s \\
& z_s = -\sum_{j=1}^{n} R_{sj} x_j, \quad s = 1, \ldots, S \\
& v_s \geq z_s - \mu, \quad s = 1, \ldots, S \\
& v_s \geq 0, \quad s = 1, \ldots, S \\
& x \in X_0
\end{aligned}
$$

Basket of 719 Securities. Mean–Semideviation Model



Key Requirement: Monotonicity

$$\rho(Z) = \mathbb{E}[Z] + \kappa\, r[Z]$$

Consistency with Stochastic Dominance (Ogryczak–R., 1997)

$$\mathbb{E}[u(Z)] \le \mathbb{E}[u(W)],\ \forall \text{ nondecreasing and convex } u(\cdot) \Rightarrow\ \rho[Z] \le \rho[W]$$

Consistency with Pointwise Order (Artzner *et. al.*, 1999)

$$Z \ \le\ W \text{ a.s. } \ \Rightarrow\ \rho[Z] \le \rho[W]$$

Mean–semideviation and mean–deviation from quantile models are consistent for $0 \le \kappa \le 1$, but not mean–variance.

Unique optimal solutions of consistent optimization models

$$\min_{x \in X}\ \rho(Z_x)$$

cannot be strictly dominated (in the corresponding sense)

Space of uncertain outcomes $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$

A functional $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is a coherent risk measure if it satisfies the following axioms

- Convexity: $\rho(\lambda Z + (1-\lambda)W) \le \lambda\rho(Z) + (1-\lambda)\rho(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}$
- Monotonicity: If $Z \le W$ then $\rho(Z) \le \rho(W)$, $\quad \forall\ Z, W \in \mathcal{Z}$
- Translation Equivariance: $\rho(Z + a) = \rho(Z) + a$, $\quad \forall\ Z \in \mathcal{Z}$, $a \in \mathbb{R}$
- Positive Homogeneity: $\rho(\tau Z) = \tau\rho(Z)$, $\quad \forall\ Z \in \mathcal{Z}$, $\tau \ge 0$

Kijima-Ohnishi (1993) – no monotonicity
Artzner-Delbaen-Eber-Heath (1999–) - space $\mathcal{L}_\infty$

R.-Shapiro (2005) – spaces $\mathcal{L}_p$, ...

Good news: $\mathbb{E}[Z]$ is coherent

Coherent Risk Measures

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Coherence of Mean–Semideviation

For simplicity, semideviation of order $m = 1$ with $\kappa = 1$:

$$\rho(Z) = \mathbb{E}[Z] + \mathbb{E}\Big[(Z - \mathbb{E}Z)_+\Big] = \mathbb{E}\Big\{\max\Big(\mathbb{E}[Z], Z\Big)\Big\}$$

Convexity follows from the convexity of $Z \mapsto \max\Big(\mathbb{E}[Z], Z\Big)$ a.s.

Monotonicity follows from monotonicity of $Z \mapsto \max\Big(\mathbb{E}[Z], Z\Big)$ a.s.

Translation follows from translation of $Z \mapsto \max\Big(\mathbb{E}[Z], Z\Big)$ a.s.

Pos. Homogeneity follows from pos. homogeneity of $\max\Big(\mathbb{E}[Z], Z\Big)$ a.s.

Convex combination of coherent measures of risk is coherent

$$\rho(Z) = \lambda_1\rho_1(Z) + \lambda_2\rho_2(Z) + \cdots + \lambda_L\rho_L(Z)$$
$$\lambda_1 + \lambda_2 + \cdots + \lambda_L = 1,$$
$$\lambda_1 \geq 0,\ \lambda_2 \geq 0, \ldots, \lambda_L \geq 0$$

$\rho(Z) = \mathbb{E}[Z] + \kappa\mathbb{E}\Big[(Z - \mathbb{E}Z)_+\Big]$ is coherent for $\kappa \in [0, 1]$

The Value at Risk at level $\alpha \in (0,1)$ of a random cost $Z \in \mathcal{Z}$:

$$\text{V@R}_\alpha^+(Z) \triangleq \inf\{\eta : F_Z(\eta) \geq 1-\alpha\} = F_Z^{-1}(1-\alpha)$$

Monotonicity: $Z \leq V \Longrightarrow \text{V@R}_\alpha^+(Z) \leq \text{V@R}_\alpha^+(V)$

Translation: $\text{V@R}_\alpha^+(Z+c) = \text{V@R}_\alpha^+(Z) + c$, for all $c \in \mathbb{R}$

Positive Homogeneity: $\text{V@R}_\alpha^+(\gamma Z) = \gamma\, \text{V@R}_\alpha^+(Z)$, for all $\gamma \geq 0$

However, it is not convex

Counterexample: Two independent variables

$$Z = \begin{cases} 0 & \text{with probability } 1-p \\ 1 & \text{with probability } p \end{cases} \qquad V = \begin{cases} 0 & \text{with probability } 1-p \\ 1 & \text{with probability } p \end{cases}$$

For $p < \alpha < 1$ we have $\text{V@R}_\alpha^+(Z) = \text{V@R}_\alpha^+(V) = 0$

If $p < \alpha < 1-(1-p)^2$, we have non-convexity

$$\text{V@R}_\alpha^+\big(\lambda Z + (1-\lambda)V\big) > 0 = \lambda\, \text{V@R}_\alpha^+(Z) + (1-\lambda)\, \text{V@R}_\alpha^+(V)$$

$$\mathrm{AV@R}_\alpha^+(Z) \triangleq \frac{1}{\alpha}\int_0^\alpha \mathrm{V@R}_\beta^+(Z)\, d\beta$$

If the $(1-\alpha)$-quantile of Z is unique

$$\mathrm{AV@R}_\alpha^+(Z) = \frac{1}{\alpha}\int_{\mathrm{V@R}_\alpha^+(Z)}^\infty z\, dF_Z(z) = \mathbb{E}\big[Z \,|\, Z \geq \mathrm{V@R}_\alpha^+(Z)\big]$$

Extremal representation

$$\mathrm{AV@R}_\alpha^+(Z) = \inf_{\eta\in\mathbb{R}} \left\{\eta + \frac{1}{\alpha}\mathbb{E}\big[(Z-\eta)_+\big]\right\}$$

The minimizer $\eta = \mathrm{V@R}_\alpha(Z)$

Connection to weighted deviation from α-quantile:

$$\delta_\alpha^+(Z) = \mathrm{AV@R}_\alpha^+(Z) - \mathbb{E}[Z], \quad \alpha \in [0,1].$$

Extremal representation

$$\mathrm{AV@R}_\alpha^+(Z) = \inf_{\eta\in\mathbb{R}} \left\{\eta + \frac{1}{\alpha}\mathbb{E}\Big[(Z-\eta)_+\Big]\right\}$$

Convexity follows from joint convexity in (η, Z) of $\left\{\cdots\right\}$

Monotonicity follows from monotonicity w.r.t. Z of $\left\{\cdots\right\}$

Translation follows from $\eta \leftrightarrow \eta - c$ in $\left\{\cdots\right\}$

Pos. Homogeneity follows from pos. homogeneity in (η, Z) of $\left\{\cdots\right\}$

Linear Programming Representation of AV@R

Suppose Z has finitely many realizations $z_1, z_2, \dots, z_S$ with probabilities $p_1, p_2, \dots, p_S$

$$\begin{aligned} \min_{v,\eta} \quad & \eta + \frac{1}{\alpha} \sum_{s=1}^{S} p_s v_s \\ \text{s.t.} \quad & v_s \geq z_s - \eta, \quad s = 1, \dots, S \\ & v_s \geq 0, \qquad\quad s = 1, \dots, S \end{aligned}$$

For portfolios we have to add the constraints

$$z_s = -\sum_{j=1}^{n} R_{sj} x_j, \quad s = 1, \dots, S$$

$$x \in X_0$$

and include z and x into the decision variables

Conjugate Duality of Risk Measures

Pairing of a linear topological space $\mathcal{Z}$ with a linear topological space $\mathcal{Y}$ of regular signed measures on Ω with the bilinear form

$$\langle \mu, Z \rangle = \mathbb{E}_\mu[Z] = \int_\Omega Z(\omega)\,\mu(d\omega)$$

We assume standard conditions on pairing and the polarity: $(\mathcal{Z}_+)^\circ = \mathcal{Y}_-$

Dual Representation Theorem

If $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is a lower semicontinuous* coherent risk measure, then

$$\rho(Z) = \max_{\mu \in \mathcal{A}} \int_\Omega Z(\omega)\,\mu(d\omega), \quad \forall Z \in \mathcal{Z}$$

with a convex closed $\mathcal{A} \subset \mathcal{P}$ (set of probability measures in $\mathcal{Y}$).

Delbaen (2001), Föllmer–Schied (2002), R.–Shapiro (2005),

Rockafellar–Uryasev–Zabarankin (2006), . . .

* *Lower semicontinuity is automatic if ρ is finite and $\mathcal{Z}$ is a Banach lattice*

$Z \sim V$ means that Z and V have the same distribution, $\mu_Z = \mu_V$.
$\rho : \mathcal{Z} \to \mathbb{R}$ is law invariant if $Z \sim V \Longrightarrow \rho(Z) = \rho(V)$

Kusuoka Theorem

If $(\Omega, \mathcal{F}, P)$ is atomless and $\rho : \mathcal{L}_1(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is law invariant, then

$$\rho(Z) = \sup_{m \in \mathcal{M}} \int_0^1 \text{AV@R}_\alpha^+(Z)\ m(d\alpha)$$

where $\mathcal{M}$ is a convex set of probability measures on $(0, 1]$.

Spectral measure

$$\rho(V) = \int_0^1 \text{AV@R}_\alpha^+(Z)\ m(d\alpha)$$

Spectral measures have dual utility form:

$$\rho(Z) = \int_0^1 F_Z^{-1}(\beta)\ dw(\beta)$$

Optimization of Risk Measures

"Minimize" over $x \in X$ a random outcome $Z_x(\omega) = f(x, \omega)$, $\omega \in \Omega$

Composite Optimization Problem

$$\min_{x \in X} \rho(Z_x) \qquad \text{(P)}$$

Theorem

Let $x \mapsto Z_x(\omega)$ be convex and $\rho(\cdot)$ be coherent. Suppose that $\hat{x} \in X$ is an optimal solution of (P) and $\rho(\cdot)$ is continuous at $Z_{\hat{x}}$. Then there exists a probability measure $\hat{\mu} \in \partial\rho(Z_{\hat{x}}) \subseteq \mathcal{A}$ such that $\hat{x}$ solves

$$\min_{x \in X} \mathbb{E}_{\hat{\mu}}[Z_x] = \min_{x \in X} \max_{\mu \in \mathcal{A}} \mathbb{E}_{\mu}[Z_x]$$

We also have the duality relation:

$$\min_{x \in X} \rho(Z_x) = \max_{\mu \in \mathcal{A}} \inf_{x \in X} \mathbb{E}_{\mu}[Z_x]$$

Duality in Portfolio Optimization - Game Model

Suppose the vector of return rates of assets has S realizations

R_{js} - return rate of asset $j = 1, \dots, n$ in scenario $s = 1, \dots, S$

Portfolio return (negative) in scenario s

$$Z_s(x) = -\sum_{j=1}^{n} R_{js} x_j$$

Portfolio Problem

$$\min_{x \in X} \rho\big(Z(x)\big)$$

By homogeneity, we may assume that $\sum_{j=1}^{n} x_j = 1$

Equivalent Matrix Game

$$\max_{x \in X} \min_{\mu \in \mathcal{A}} \sum_{j=1}^{n} \sum_{s=1}^{S} x_j R_{js} \mu_s$$

x - mixed strategy of the investor

μ - mixed strategy of the opponent (market)

Expected-Value Model

$$\min_{x \in X} c^T x + \mathbb{E}\Big[Q(x))\Big]$$

where $Q(x)$ is the optimal value of the random second-stage problem

$$\begin{aligned} \min\ & q^T y \\ \text{s.t. } & Tx + Wy = h, \\ & y \geq 0, \end{aligned}$$

(q, T, h) - random data of the second-stage problem

- c is deteministic
- (q, T, h) become known after the first stage

For finite scenario case - powerful decomposition methods

Two-Stage Model: Risk-Averse Version

$$\min_{x \in X} \rho_1\Big(c^T x + Q(x)\Big)$$

where $Q(x)$ is the optimal value of the second-stage problem

$$\begin{aligned} Q(x) = \min\ & \rho_2\Big(q^T y\Big) \\ \text{s.t. } & Tx + Wy = h, \\ & y \geq 0, \end{aligned}$$

and $\xi = (q, T, h)$ - random data of the second-stage problem

- c is random
- (T, h) become known after the first stage
- q may be still unknown (conditional distribution)

Risk-Averse Two-and-Half-Stage Model. Finite Scenario Case

Second-stage scenarios: c_s, T_s, h_s, $s = 1, \ldots, S$
Final scenarios: q_{sj}, $j \in J(s)$

$$\min_{x \in X} \rho_1\Big(c^T x + Q(x)\Big)$$

where $Q(x)$ is the optimal value of the second-stage problem;
In scenario s its value is

$$\begin{aligned} Q_s(x) = \min\ & \rho_{2s}\big(q_s^T y\big) \\ \text{s.t. } & T_s x + W y = h_s, \\ & y \geq 0, \end{aligned}$$

q_s is random and has realizations q_{sj}, $j \in J(s)$

This structure of the problem follows from the general theory of dynamic measures of risk (lecture tomorrow)

Dual Representation of the Two-Stage Problem

Risk-averse first-stage problem

$$\min_{x\in X} \max_{\mu\in\mathcal{A}} \sum_{s=1}^{S} \mu_s\Big[c_s^T x + Q_s(x)\Big]$$

Risk-averse second-stage problem

$$\begin{aligned} Q_s(x) = \min_{y} \max_{\nu\in\mathcal{B}_s} & \sum_{j\in J(s)} \nu_j q_j^T y \\ \text{s.t. } & T_s x + W_s y = h_s \qquad \text{(multipliers } \pi_s\text{)} \\ & y \geq 0 \end{aligned}$$

The sets of probability measures:

$$\mathcal{A} = \partial\rho_1(0)$$
$$\mathcal{B}_s = \partial\rho_{2s}(0)$$

Z_x - random outcome (*e.g.*, cost)
Y - benchmark random outcome, e.g. $Y(\omega) = Z_{\bar{x}}(\omega)$ for some $\bar{x} \in X$

New Model

$$\begin{aligned} &\min \mathbb{E}[Z_x] && \text{(or some other objective)} \\ &\text{subject to } Z_x \preceq_{\mathcal{U}} Y && \text{(stochastic ordering constraint)} \\ &x \in X \end{aligned}$$

Z_x is preferred over Y by all decision makers having disutility functions in the generator $\mathcal{U}$:

$$\mathbb{E}[u(Z_x)] \le \mathbb{E}[u(Y)] \quad \forall\ u \in \mathcal{U}$$

All nondecreasing $u(\cdot)$ - first order stochastic dominance $\preceq_{\text{st}}$
All nondecreasing convex $u(\cdot)$ - increasing convex order $\preceq_{\text{icx}}$

$$\min \mathbb{E}[Z_x]$$
$$\text{subject to } Z_x \preceq_{\text{icx}} Y$$
$$x \in X$$

X - convex set in $\mathcal{X}$ (separable locally convex Hausdorff vector space)
$x \mapsto Z_x$ is a continuous operator from $\mathcal{X}$ to $\mathcal{L}_1(\Omega, \mathcal{F}, P)$
$x \mapsto Z_x(\omega)$ is convex for P-almost all $\omega \in \Omega$

Primal: $\mathbb{E}[u(Z_x)] \le \mathbb{E}[u(Y)]$ for all convex nondecreasing $u : \mathbb{R} \to \mathbb{R}$
Inverse: $\int_0^1 F_{Z_x}^{-1}(p)\, dw(p) \le \int_0^1 F_Y^{-1}(p)\, dw(p)$ for all convex nondecreasing $w : [0,1] \to \mathbb{R}$

Main Results

- Utility functions $u : \mathbb{R} \to \mathbb{R}$ and rank dependent utility functions $w : [0,1] \to \mathbb{R}$ play the roles of Lagrange multipliers
- Expected utility models and rank dependent utility models are Lagrangian relaxations of the problem

Lagrangian in Direct Form

$$L(x,u) = \mathbb{E}\Big[Z_x + u(Z_x) - u(Y)\Big]$$

$u(\cdot)$ - convex function on $\mathbb{R}$

Theorem

Assume Uniform Dominance Condition (a form of Slater constraint qualification). If $\hat{x}$ is an optimal solution of the problem then there exists a function $\hat{u} \in \mathcal{U}$ such that

$$L(\hat{x},\hat{u}) = \min_{x \in X} L(x,\hat{u}) \tag{1}$$

$$\mathbb{E}\Big[\hat{u}(Z_{\hat{x}})\Big] = \mathbb{E}\Big[\hat{u}(Y)\Big] \tag{2}$$

Conversely, if for some function $\hat{u} \in \mathcal{U}$ an optimal solution $\hat{x}$ of (1) satisfies the dominance constraint and (2), then $\hat{x}$ is optimal

Implied Rank Utility (Distortion) Function

Lagrangian in Inverse Form

$$\Phi(x, w) = \int_0^1 F_{Z_x}^{-1}(p)\, d(p + w(p)) - \int_0^1 F_Y^{-1}(p)\, dw(p)$$

$w(\cdot)$ - convex function on $[0, 1]$

Theorem

Assume Uniform Dominance Condition (a form of Slater constraint qualification). If $\hat{x}$ is an optimal solution of the problem, then there exists a function $\hat{w} \in \mathcal{W}$ such that

$$\Phi(\hat{x}, \hat{w}) = \min_{x \in X} \Phi(x, \hat{w}) \tag{3}$$

$$\int_0^1 F_{Z_{\hat{x}}}^{-1}(p)\, d\hat{w}(p) = \int_0^1 F_Y^{-1}(p)\, d\hat{w}(p) \tag{4}$$

If for some $\hat{w} \in \mathcal{W}$ an optimal solution $\hat{x}$ of (3) satisfies the inverse dominance constraint and (4), then $\hat{x}$ is optimal